

## Linear Relations

### 8.1 Special case

Some years ago in Honolulu, the members of the city council announced that they had managed to lower the property tax rate from \$7 per \$1000 of the assessed value of the property to \$6 per \$1000. However, they did not disclose that they also raised the assessed value of a property from 70% of the market value of the property to 100% of the market value. A question naturally arose: What was the net effect of the "tax reduction"? To answer the question, we have to know the precise relationship between the amount of the tax a property owner pays and the two quantities, the tax rate and the market value of the property.

This is one of the ways in which the necessity of determining the relationship between two or more quantities arises in our daily lives and in mathematics. Such a relationship is called a **functional relationship**, and for the above example, we say that the amount of the tax is a **function** of the tax rate and the market value of the property. The quantities over which we have some control, like the tax rate and the market value, are called **independent variables** while the quantity determined by the independent variables (the amount of the property tax, in this case) is called the **dependent variable**.

When the functional relationship between two quantities  $x$  and  $y$  is given by an equation of the form  $y = mx + b$  or  $y = m(x - a) + c$ , the relationship is called "linear" and we say that  $x$  and  $y$  are linearly related. We will study such relationship.

We first examine the significance of the constants  $m$  and  $b$  in the equation . To get a concrete picture, let us look at an example, , and make a table of values for  $x$  and  $y$ , and plot the graph. (Make the table for yourself by copying the values of  $x$  and then computing the corresponding values of  $y$ ). We obtain the table given on the right:

x	y
0	3
1	5
2	7
3	9

2.

4	11
5	13

Table 1.1

We make the following observations:

- (I) The 3 in is the value of y when . In fact, it is readily seen that if the relationship between x and y is given by the equation , the b in is the value of y when .
- (II) As we increase the value of x by one unit, the value of y increases by 2 units (or more precisely, the value of y changes by units). In general, if the relationship between x and y is given by the equation , then as we increase the value of x by 1 unit, the value of y changes by m units (increases or decreases depending on the algebraic sign of m). That is, m, the coefficient of x, is the rate of change of y with respect to x.

These observations enable us to obtain the equation from the table that describes the linear relationship. For example, suppose that the relationship between x and y is given by Table 1.2:

x	y
0	13
1	17
2	21
3	25
4	29

Table 1.2

We see that when , , and that the rate of change of y with respect to x is 4 (that is, as x increases by 1 unit, y changes by 4 units). Therefore, the equation that describes the relationship between x and y is .

Let us get back to the first example. When we plot the points given in Table 1.1 on a graph paper, we will notice that all the points seem to lie on a straight line.. In fact, if we plot more points, even assigning fractional values to x, we will see that they all lie on the same straight line. The totality of the points whose coordinates satisfy the equation is called the **graph of the equation**. The graph of the equation is shown in Fig. 1.3.



4.

Fig. 1.3

In this case, the graph of the equation is a straight line. In fact, it turns out that the graph of an equation of the type  $y = mx + b$  is a straight line. It is for this reason that the equation of the type  $y = mx + b$  is called a **linear equation**, and we say that  $x$  and  $y$  are **linearly related** or that  $y$  is a **linear function** of  $x$ .

**Exercises 8.1**

1. Find the corresponding values of  $y$  to complete the table in each case and then plot the graph of the equation:

(a)

x	y
-----	
0	
1	
2	
3	
4	

(b)

x	y
-----	
0	
1	
2	
3	
4	

(c)

x	y
-----	
0	
1	
2	
4	
6	

(d)

x	y
-----	
0	
4	
8	
12	
16	

2. In each case find the equation that describes the relationship shown in the table:

(a)

x	y
-----	
0	23
1	28
2	33
3	38
4	43

(b)

t	d
-----	
0	15
1	11
2	7
3	3
4	-1

(c)

p	q
-----	
0	-5.0
4	-3.5
8	-2.0
12	-0.5
16	1.0

6.

(d)		(e)		(f)	
x	y	w	z	t	s
0	17	0.0	1.50	0	45
5	19	0.2	1.65	10	43
10	21	0.4	1.80	20	41
15	23	0.6	1.95	30	39
20	25	0.8	2.10	40	37

3. Starr Car Rental Company rents out cars at \$30 per day and 15¢ per mile.

Let  $x$  = the number of miles a customer drives in one day, and  
 $y$  = the cost (in dollars) for the day.

(a) Fill in the table:

x	y
0	30
1	
2	
3	
4	
5	

(b) Find the equation that describes the relationship between  $x$  and  $y$ . (Make sure that your equation gives the values of  $y$  that you entered in the table.)

(c) If a customer drives 80 miles, what is the cost of renting the car for the day?

(d) If a businessman has \$72 for a travel allowance and if he rents a car from the company for one day, how many miles can he drive without exceeding the travel allowance?

4. The 1990 Federal Income Tax Schedule for those filing as "Single" said:

If your taxable income does not exceed \$19,450 , then your tax is 15% of your taxable income.

Let  $x$  = the taxable income (in dollars) of a person filing as "Single", (and whose taxable income did not exceed \$19,450), and

$T$  = the tax (in dollars) the person paid.

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(a) Fill in the table:

x	T
0	
100	
200	
300	
400	

(b) Find the equation by which you can compute the tax, given the taxable income in the range indicated.

The following questions refer to persons who had filed as "Single" for the year 1990.

(c) If Tom's taxable income was \$7,500, what was his tax?

(d) If Maria paid \$2,451 for the federal income tax, what was her taxable income?

5. The formula gives the relationship between Fahrenheit and Centigrade temperature measures. A quick rule of thumb for converting from Centigrade to Fahrenheit measure is "double the Centigrade temperature and add 30".

(a) Write the equation for the rule of thumb conversion procedure, using  $F$  for the approximate Fahrenheit measure and  $C$  for the Centigrade measure.

(b) Graph both equations on the same graph paper and label your graphs as "true eq." and "rule-of-thumb eq.". (Make tables for the equations.)

(c) Eventually which formula gives higher Fahrenheit temperature? (Give the reason for your answer.)

(d) At what Centigrade temperature do the two formulas give the same Fahrenheit temperature?

- (e) Let us call the approximate temperature obtained by the rule-of-thumb formula "acceptable" if the approximate temperature differs from the true temperature by at most 3 degrees Fahrenheit. Then, for what range of the centigrade temperatures does the rule-of-thumb formula give the "acceptable" values?

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6. Economy Rent-A-Car Company (we call it E-Company) rents out cars at \$10 per day and 25¢ per mile while Jet-Age Car Rental Company (call it J-Company) rents out cars at \$30 per day and 15¢ per mile.

Let  $x$  = the number of miles a customer drives,

= the cost of renting a car from E- Company and driving  $x$  miles,

= the cost of renting a car from J-Company and driving  $x$  miles.

- (a) Fill in the tables:

$x$	$x$
0	0
10	10
20	20
30	30
40	40
50	50

- (b) Find the equations that express the relationships between and  $x$  and between and  $x$ .
- (c) If a tourist expects to drive about 90 miles in one day, which company gives a lower cost?
- (d) Up to how many miles of driving, is renting a car from E-Company cheaper than renting a car from J-Company?

7. When sodium carbonate is dissolved in water, the gas carbon dioxide is generated. The amount of gas generated is measured by measuring the pressure exerted by the gas. The more the sodium carbonate there is, the more gas is generated and the greater the pressure exerted by the gas. We want to determine the relationship between the amount of sodium carbonate and the pressure exerted by the resulting gas. The following is an experimental result of a chemistry student:

W (grams)	P (cm of mercury)
0.00	0.40
0.10	0.90
0.25	2.15
0.40	2.90
0.55	4.00
0.70	4.90

- (a) Plot the points on a graph paper.
- (b) Draw the line that seems to "fit the data the best".
- (c) Find the equation of the "best fitting line" that you drew in (b). (Locate two points on the line, and use their coordinates to find the equation.)
- (d) Using the equation that you obtained in (c), compute the pressure generated by a sample that contains 0.62 grams of sodium carbonate. Compare the value with the value that you can read off the graph of (b).
- (b) An unknown sample is known to contain sodium carbonate and no other gas producing substance. When the sample is dissolved in water, the pressure exerted by the resulting gas turns out to be 4.25 cm of mercury. Find the amount of sodium carbonate present in the sample.

12.

## 8.2 The general form

A more useful form of the equation describing a linear relation is given by  $y = m(x - a) + c$ . Again, let us see the significance of the constants  $a$ ,  $c$ , and  $m$  by looking at an example, say . We see that it does not require any computation to find the value of  $y$  when , and that when the values of  $x$  are near 15, the values of  $y$  are near 8. So, we make the table of values starting at :

x	y
15	8
16	11
17	14
18	17
19	20

Table 1.5

We see that in the general case  $y = m(x - a) + c$ , the  $c$  is the value of  $y$  when  $x = a$ , and that the significance of  $m$  is the same as in the previous case, namely that  $m$  is the rate of change of  $y$  with respect to  $x$ . Again one of the reasons why we examine the significance of the constants  $a$ ,  $c$ , and  $m$  in the equation  $y = m(x - a) + c$  is that we want to be able to obtain the equation of the linear relation when the relationship is given by a table. For example, suppose that the relationship between two quantities  $x$  and  $y$  is given by Table 1.6.

x	y
40	17
45	20
50	23
55	26
60	29

Table 1.6

We see that as  $x$  increases by 5 units,  $y$  increases by 3 units, so that the per unit increase of  $y$  with respect to  $x$  is  $3/5$ . This is the value of  $m$ . So the equation that describes the relationship shown in the table is . It is always a good practice to check whether this equation does

give the rest of the values in the table by evaluating the expression at one more value of  $x$ , say at  $x = 45$ . We see that

$$\begin{aligned} &= \\ &= 3 + 17 \\ &= 20 \end{aligned}$$

Thus it does give the next pair  $(45,20)$ , and in fact we can visualize that the equation does give the rest of the values.

We note that there are various ways of writing the equation describing the relationship given in Table 1.6. We can use any of the pairs,  $(40,17)$ ,  $(45,20)$ ,  $(50,23)$ , etc., as the "reference point" to write the equations:

and so on.

All of these equations can be reduced to  $y = 3x + 17$ .

We want to mention here the commonly used geometrical terms associated with the equations  $y = mx + b$  and  $y - y_1 = m(x - x_1)$ . The coefficient  $m$  of  $x$  is called the **slope** of the line and  $b$  in the first equation is called the **y-intercept** of the line.

14.

## Exercises 8.2

1. Find the corresponding values of  $y$  to complete the table in each case and then sketch the graph of the equation:

(a)

$xy$	$x$
50	
51	
52	
53	
54	

(b)

$y$
70
71
72
73
74

(c)

$xy$	$x$
-4	
-3	
-2	
0	
2	

(d)

$y$
6
11
16
21
26

2. In each case find the equation that describes the relationship shown in the table:

(a)

$x$	$y$
25	17
26	21
27	25
28	29
29	33

(b)

$x$	$y$
50	28
55	31
60	34
65	37
70	40

(c)

$u$	$v$
19	50
22	49
25	48
28	47
31	46



16.

(d)		(e)		(f)	
s	t	p	q	t	w
600	37	1970	11,000	3.5	1.0
650	40	1974	12,500	3.7	2.3
700	43	1978	14,000	3.9	3.6
750	46	1982	15,500	4.1	4.9
800	49	1986	17,000	4.3	6.2

3. The 1990 Federal Income Tax Schedule for those filing as "Single" said:

If your taxable income is over \$19,450 but not over \$47,050, then your tax is \$2,917.50 plus 28% of the amount over \$19,450.

Let  $x$  = the taxable income (in dollars) of a person filing as "Single", (and whose taxable income was between \$19,450 and \$47,050), and

$T$  = the tax (in dollars) the person paid.

(a) Fill in the table:

x	T
19,450	2917.50
19,550	
19,650	
19,750	
19,850	

(b) Find the equation by which you can compute the tax, given the taxable income in the range indicated.

The following questions refer to persons who had filed as "Single" for the year 1990.

(c) If Jim's taxable income was \$24,300, what was his tax?

(d) If Jill paid \$5,213.50 for the federal income tax, what was her taxable income?

4. A salesman at a certain company has the option of choosing one of two plans according to which they get paid per month. The two plans are the following:

Plan 1: This plan pays the worker \$600 plus the commission that is 2% of the amount over \$10,000 of his sales (for the month).

Plan 2: This plan is strictly by commission and he is paid 4.5% of his total sales for the month.

Let  $x$  be the amount (in dollars) of a worker's sales for a month.

Let  $y_1$  be a worker's pay for the month according to Plan 1, and  $y_2$  be a worker's pay for the month according to Plan 2.

- (a) In both plans, assume that a worker's sales is at least \$10,000, and fill out the tables:

$x$	$x$
10,000	10,000
10,100	10,100
10,200	10,200
10,300	10,300
10,400	10,400

- (b) Find the equations that express the relationships between  $x$  and  $y_1$  and between  $x$  and  $y_2$ .
- (c) If in the month of March a worker had a total sales of \$15,000 to be paid by Plan 1, what was his pay for the month?
- (d) If another worker had the total sales of \$15,000 in the month of March and chose to be paid by Plan 2, what was his pay for the month?
- (e) From what total sales on does Plan 2 give a higher pay than Plan 1?

18.

5. In a certain city the real estate property is taxed (annually) at the rate of \$12 per \$1000 on the amount that is the assessed value of the property less \$50,000 exemption.

Let  $V$  = the assessed value of a property, and  
 $T$  = the tax on the property.

(a) Fill out the table below:

$V$	$T$
50,000	0
51,000	
52,000	
53,000	

- (b) Find the formula that expresses the tax  $T$  in terms of the assessed value  $V$  of a property. (Assume that  $V$  is at least \$50,000.)
- (c) If a property is assessed at \$180,000 , what is the tax for the property?
- (d) If a home owner paid \$2100 in the real estate tax last year, what was the assessed value of his property?