QUADRATIC EQUATIONS

Definition: A *quadratic* equation is an equation involving an unknown to the second power (degree), but no higher power. The general form of the quadratic equation is

$$ax^2 + bx + c = 0, \text{ where x is the unknown and } a \neq 0.$$

The coefficient of the unknown of the highest exponent is called the **leading coefficient.** For example, the leading coefficient of the expression $7x^2 - 5x + 4$ is 7, while the leading coefficient of the expression $10 + 3u - 5u^2$ is -5.

Our first inclination is to think of *quadratic* as dealing with "four". But the dictionary's definition of quadratic is "containing quantities of the second degree". The name comes from "quadrature", an ancient problem of constructing a square having the same area as the given figure using only a compass and a straight edge. (To see that quadrature is not such a simple problem, try constructing a square having the same area as the rectangle whose length is 5 inches and whose width is 2 inches, using only a compass and a straight edge.)

7.1 Solving Quadratic Equations by Factoring

To solve quadratic equations by factoring:

a) write the equation in standard form (bring all terms to the left-hand side so that the right-hand side is zero),

b) factor the expression on the left (or right) side,

c) solve for the values of the unknown using the multiplication property of zero.

Multiplication Property of Zero:
If A B = 0, then either A = 0 or B = 0. There are no other possibilities.
But if A B = 1 (or any other non-zero number), there are infinite possibilities. For example,

A = 1, B = 1

etc.

Example 1: Find the solutions of the equation $2x^2 = 5x + 3$.

2.

solution: $2x^2 - 5x - 3 = 0$ bring all terms to the left-hand side so that the right-hand side is zero

$(2x + 1)(x - 3) = 0$ factor the trinomial

$2x + 1 = 0$ or $x - 3 = 0$ multiplication property of zero

$2x = - 1$

So, and 3 are the solutions. [Note: Equation of degree 2 has , in general, 2 solutions.]

Example 2: Find the solutions of the equation $12n^2 = 9n$.

solution: $12n^2 - 9n = 0$ write equation in standard form

$3n(4n - 3) = 0$ factor out common factors

$3n = 0$ or $4n - 3 = 0$ set each factor equal to zero

The solutions are 0 and .

Example 3: Find the solutions: $y^2 - 16 = 0$

solution: $y^2 - 16 = 0$ the right-hand side is already equal to zero

$(y + 4)(y - 4) = 0$ factor the difference of squares

$y + 4 = 0$ or $y - 4 = 0$

$y = - 4$ or $y = 4$

The solutions are -4 and 4, which can also be written as .

Chapter 7

Exercises 7.1

Solve the following quadratic equations by factoring:

(a) $x^2 - 7x + 6 = 0$

(b) $n^2 + n = 6$

(c) $y^2 + 2y = 0$

(d) $3m = m^2$

(e) $9t^2 - 1 = 0$

(f) $4g^2 = 25$

(g) $2a^2 = 5a - 3$

(h) $3 + w = 2w^2$

(i) $12x^2 = 27$

(j) $12x^2 = 27x$

(k) $2n^2 + 10 = 12n$

(l) $6y^2 = 6 - 5y$

(m) $(a - 3)(2a + 1) = 39$

(n) $3p(2p - 1) = 18$

(o) $(4m - 3)(m - 2) = 6$

(p) $(2t + 5)(8t - 5) = 30t$

(q) $6g^2 + 56 = 37g$

(r) $1 = 64K^2$

(u) $0.09n^2 = 36$

(v) $y^2 - 20 = y$

7.2 Solving Quadratic Equations by Extracting Roots

Recall the earlier section on the Pythagorean Theorem: $a^2 + b^2 = c^2$. If a = 3 and b = 4, then

We took the square root (extracted the root) of 25 to get 5. But is 5 the only answer? If , then c = 5 or c = - 5 (since after squaring a positive or a negative number, we always get a positive value). For the problem above we were interested in the <u>length</u> of side c, therefore, the positive value is automatically chosen. In this section, we find ALL possible solutions--both positive and negative.

Equations of a more general forms $x^2 - c = 0$ or $(ax - n)^2 - c = 0$, where c is a positive number, can be solved by root extraction. Actually this method is an extension of the method covered in the preceding section.

We have been saying that the expression is not factorable. The statement is not entirely correct. We should say that is not factorable **using only integers**. If we allow numbers like square roots of positive numbers to be used, then the expression becomes factorable because 3 can be written as , and so we can factor as

=

=

So, if we had the problem of finding all solutions of the equation , we bring all terms to the left-hand side and factor the left-hand side, as in the preceding section:

= 0

The solutions are and , which can be written as .

It is easy to see that we can replace 3 with any positive number. So, we have the following result:

For any positive number a, if , then .

Example 1: Find the solutions: $y^2 - 16 = 0$

solution: isolate the squared variable

The solutions are .

Example 2: Solve the equation: $12n^2 - 9 = 0$

solution: $12n^2 = 9$ isolate squared term

The solutions are and , which can also be written as and .

Example 3: Solve the equation: $3(x - 1)^2 = 24$

solution: $3(x - 1)^2 = 24$ squared quantity is isolated

$(x - 1)^2 = 8$

The solutions are and , which can also be written as and .

"Why can't we use extraction of roots for other forms of the quadratic equation?"

$ax^2 + bx + c = 0$: $2x^2 + 3x + 1 = 0$

$2x^2 = - 3x - 1$

6.

$ax^2 + bx = 0$: $x^2 - 5x = 0$

$x^2 = 5x$

Exercises 7.2

Solve the following quadratic equations by extracting roots:

(a) $x^2 - 4 = 0$

(b) $n^2 = 25$

(c) $p^2 - 2 = 0$

(d) $m^2 = 7$

(e) $a^2 - 12 = 0$

(f) $K^2 = 27$

(g) $2y^2 - 98 = 0$

(h) $3z^2 = 48$

(i) $3w^2 = 30$

(j) $2t^2 - 126 = 0$

(k) $5x^2 = 1$

(l) $2a^2 - 7 = 0$

(m) $4p^2 = 1$

(n) $25m^2 = 9$

(o) $12k^2 - 5 = 0$

(p) $24z^2 = 8$

(u) $2(7w + 3)^2 = 128$

(v) $3(2y - 5)^2 = 24$

7.3 Solving Quadratic Equations by the Quadratic Formula

The *quadratic formula* can be derived by solving the general quadratic equation
$ax^2 + bx + c = 0$ using the method known as completing the square (which is not discussed here).

When a procedure is often repeated, mathematicians prefer to use a formula to shorten the number of steps. Use of the quadratic formula produces results more quickly than by the method of completing the square. In using the formula, you want to:

a) write the equation in standard form,
b) correctly identify a, b, and c,
c) carefully substitute into the formula.

Example 1: Find the solutions of the equation $x^2 + 4x - 6 = 0$

Solution: $x^2 + 4x - 6 = 0$ equation is in standard form; hence, $a = 1$, $b = 4$, $c = - 6$

Therefore, .

Example 2: Find the solutions of the equation: $2n^2 = n + 3$

Solution: $2n^2 - n - 3 = 0$ write the equation in standard form and substitute into the formula

[Note: When the solutions are rational (as in this case), the equation could have been solved by factoring. Try this and see if you obtain the same results.]

Example 3: Solve the equation: $2 + 2w + w^2 = 0$

Solution: $w^2 + 2w + 2 = 0$ write the equation in standard form and

substitute into the formula

But, since the square root of - 4 is not a real number we say that **there are no real solutions**.

In the previous examples, we used the formula with equations whose left-hand side contained a trinomial expression. What if the left-hand side is a quadratic binomial?

Example 4: Solve the equation: $8x^2 + 4x = 0$

Solution: If we wish to use the quadratic formula, then a = 8, b = 4, and c = 0.

But since the left-hand expression is easily factorable, there is no need to use the formula.

$4x(2x + 1) = 0$

10.

$$4x = 0 \quad \text{or} \quad 2x + 1 = 0$$

<u>Example 5</u>: Solve the equation: $12y^2 - 9 = 0$

Solution: If we wish to use the quadratic formula, then $a = 12$, $b = 0$, and $c = -9$.

But since we can use the extraction of roots method, there is no need to use the formula.

Although the quadratic formula can be used to solve all quadratic equations, it is often the most time-consuming. We usually reserve the use of the quadratic formula for those situations when factoring or extracting roots do not work.

Exercises 7.3

Solve the following quadratic equations by the most effective method.

(a) $x^2 - 8x + 15 = 0$

(b) $2n^2 + 3n = 2$

(c) $6y^2 - 14y = 0$

(d) $6z^2 = 48$

(e) $2a^2 = 3a + 1$

(f) $m^2 + 10m = 4$

(i) $n^2 + 8n = 1$

(j) $2z^2 = 3z - 8$

(k) $4a^2 + 5a = 0$

(l) $3m^2 + 4m = 3$

(m) $4t^2 = 4t + 7$

(n) $12w^2 - 27 = 0$

(o) $x^2 - 2x = 5 + 2x$

(p) $2y^2 + 2y - 3 = 5 - y^2$

(q) $(n + 1)^2 = 2 - 4n$

(r) $(2x - 1)^2 = 12$

(s) $(2x + 1)^2 = 2 + 8x$

(t) $2(w - 1)^2 = 7w$