Some Geometric Problems

6.1 Areas of common figures

The unit of measure of an area is square units. For example, if the length is measured in feet, then the unit of measure for the area is square feet. If, for example, the length of a rectangular room is 27 feet and the width of the room is 20 feet, then the area of the room is square feet, meaning that 540 one-foot-by-one-foot squares can be laid out side by side on the floor without overlapping. If the length and the width of a rectangle are whole numbers, we have no trouble interpreting the area of a rectangle. If, on the other hand, the length or the width of a rectangle is a fraction of a unit or worse an "irrational number" of times the unit of measure of the length, the interpretation becomes troublesome. This is the reason why the area of a rectangle is defined to be the product of length and the width. We state this principle:

> **The area A of a rectangle of length L and width W is given by**
>
> .

It is of course understood that the length and the width are measured in the same units.

Example: If the dimensions (that is, the length and the width) of the floor of a room are feet by feet, how many 1 foot-by-1 foot square tiles are required to cover the floor of the room?

Solution: The area of the floor is square feet.
So, 197 whole tiles and a fraction of a tile are needed to cover the floor, and so 198 tiles are required.

Starting from the areas of rectangles, we can find the formulas by which we can compute the areas of such figures as parallelograms, triangles, and trapezoids. Let us begin with parallelograms. A parallelogram is a 4-sided figure with two pairs of parallel sides. A typical parallelogram is shown in the figure below.

To find the area of the parallelogram, we drop the perpendicular from the upper left vertex (corner) to the lower side, as shown in the figure below:

Cut out the triangular section by cutting along the perpendicular, and place the triangular section to the other side of the parallelogram as shown in the next figure:

The resulting figure is a rectangle having the length the same as the base of the parallelogram and the width equal to the height (that is, the perpendicular distance between the upper and lower bases). So, we have the following result:

The area A of a parallelogram having the base b and height h is

A = bh

To obtain the formula for the area of a triangle with base b and height h, as shown in the figure,

we cut out an exact copy of the given triangle, rotate it by **,** and place it on the given triangle, as shown in the figure below.

The resulting figure is a parallelogram having the same base as the triangle and the same height. So, if b and h are the base and the height of the triangle, respectively, and A is the area of the triangle, then

or

We state the result:

The area A of a triangle with the base b and the height h is given by

.

We point out that the height may lie "outside" of the base as in the following figure:

A less familiar figure is a **trapezoid**. A trapezoid is a 4-sided figure (a quadrilateral) with a pair of parallel sides. A typical trapezoid is shown in the figure below:

The parallel sides are called the **bases** of the trapezoid and the **perpendicular distance between the bases** is called the **height** of the trapezoid.

Just as in the case of the derivation of the formula for the area of a triangle, we cut out an exact copy of the given trapezoid, rotate it by ,
and place it on the given trapezoid, as shown in the figure below:

The resulting figure is a parallelogram whose base is the sum of the bases of the trapezoid and whose height is the same as the height of the trapezoid. So, if A is the area of the trapezoid, and a and b are the bases and h is the height of the trapezoid, then

or

We state the result:

The area A of a trapezoid with bases a and b and the height h is given by

.

There is one more figure that we want to consider --- circle.

To get an idea about the circumference and the area of a circle with radius r, we circumscribe the circle with a square as shown in the figure:

We see that the perimeter of the square is , and so the circumference of the circle is less than since going around the circle, we "cut the corners". It turns out to be closer to than . in fact, this constant by which is to be multiplied to get the circumference of the circle is defined to be , whose approximate value is . We state this definition:

The circumference C of a circle of radius r is given by

We now consider the computation of the area of a circle. Using the same figure as above, the area of the circumscribed square is . So again the area of the circle is less than and in fact it is closer to than . What is remarkable is that the constant by which we have to multiply to get the area of the circle turns out to be also . We give below the marvelous idea of Kepler who showed why the area of a circle of radius r is equal to , using modern notation.

We divide a circle of radius r into great many sectors so that each sector is so narrow that it resembles an isosceles triangle, as shown in the figure below.

If we denote area of the sector by (read as delta A, meaning a very tiny part of the area A) and the part of the circumference by (read as delta C , meaning a very tiny part of the circumference C), then we have

Summing up all these tiny pieces, we have

When we cut up the circle into very many, many sectors so that each sector is practically an isosceles triangle, then "in the limit" we have

Now, the circumference C is given by . So, replacing C by , we have

We state this wonderful result:

The area A of a circle of radius r is given by

.

Exercises 6.1

1. Compute the area of each of the following figures. (Note that more than enough information is given in each case.)

2. In the figure, ; ; and (The notation means the length of the line segment BD.)

Find the area of in two ways:

(a) By subtracting the area of from the area of ;

(b) by applying the formula for the area of a triangle directly.

3. In the figure, M is the midpoint of the base AB.

(a) If and , compute the areas of the two triangles, and .

(b) If and , find the formulas for the areas of the two triangles, and .

4. In the figure below, the square ABCD of side 10 feet is given. H and E are the midpoints of the sides AD and AB, respectively, and and

(a) Compute the areas of the four triangles, , , , and .

(b) Compute the area of the inner figure EFGH.

5. In the figure below, ABCD is a rectangle with and Points E, F, G, and H are the midpoints of the sides. Compute the area of the inner figure EFGH.

6. In the figure below, the quadrilateral ABCD is a trapezoid with sides AB and CD being parallel. The side AD is perpendicular to the bases AB and DC.

(a) If , , and , compute the area of the triangle CEB.

(b) Obtain the area of the trapezoid ABCD by adding the areas of the triangle CEB and the rectangle AECD. (Simplify the formula.)

7. Lines and are parallel lines and figures ABCD and ABEF are parallelograms. Which parallelogram has the larger area? Justify your answer.

8. Lines and are parallel lines and triangles ABC and ABD are drawn as shown.

Which triangle has the larger area? Justify your answer.

9. In the figure below, the radius of the outer circle is 8 feet and the radius of the inner circle is 4 feet..

(a) Find the areas of the two circles.

(b) The area of the bigger circle is how many times the area of the smaller circle?

(c) Find the area of the region between the two circles.

10. In the figure below, AB and CD are mutually perpendicular diameters of the circle. The diameter of the circle is 15 feet..

(a) Compute the area of the quadrilateral ACBD.

(b) Compute the area of the region inside the circle but outside of the quadrilateral ACBD.

11. In the figure below, the diameter of the larger circle is 10 feet and the diameter of each of the two smaller circles is 5 feet.

(a) Compute the area of one of the two smaller circles.

(b) The area of the smaller circle is what fractional part of the area of the larger circle?

(c) Compute the area of the region inside the larger circle but outside of the two smaller circles.

12. In the figure below, the diameter of the largest circle is 10 feet, the diameter of the second largest circle is 5 feet and the diameter of one of the two smallest circles is feet.

(a) Find the area of each of the four circles.

(b) The area of the smallest circle is what fractional part of the area of the largest circle?

(c) Find the area of the region inside the largest circle but outside the three smaller circles.

13. In the figure below, parallelogram ABCD is given. The bases AB and DC are extended, and a line perpendicular to the extensions is drawn, meeting the extensions in P and E. Then, lines parallel to the base AB are drawn meeting the sides of the parallelogram and the perpendicular line at G, F, L, I, H, M, K, J, and N.

(a) If , , , , and , compute the areas of the four small parallelograms, ABFG, GFHI, IHJK , and KJCD.

(b) Compute the area of the parallelogram ABCD in two ways:
(i) By computing the sum of the four parallelograms;

(ii) by applying the formula for the area of a parallelogram directly.

(c) If , , , , , and , compute the area of the parallelogram by adding up the areas of the four parallelograms.

Many interesting problems require some knowledge of plane geometry, particularly the properties of parallel lines and similar triangles. We will describe briefly those properties.

The first is so-called "parallel postulate", which we can regard as the definition of parallel lines.

Parallel Postulate: When two lines and are cut by a third line, if the sum of the interior angles and is equal to two right angles (or), the two lines do not intersect however far they are extended (that is, the two lines are parallel).

When two parallel lines are cut by a third line, some of the angles formed are equal. In particular, in the figure below, is equal to . These two angles are called "**alternate interior angles**". We state the result because of its importance.

When two parallel lines are cut by a third line, the alternate interior angles are equal.

This postulate has far reaching consequences. One of the consequences is that the sum of the internal angles of a triangle is always equal to two right angles or . We will show why this is so.

Given a triangle ABC, through the vertex C draw a line parallel to the opposite side, as shown in the figure below.

Then we can shift the lower angles at A and B to the upper side by making use of the "alternate interior angles" property of parallel lines given above, and so or .

Since a quadrilateral (a 4-sided figure) can be divided into two triangles by drawing a diagonal, the sum of the internal angles of a quadrilateral is . Similarly, we can show that the sum of the internal angles of a pentagon is ; the sum of the internal angles of a hexagon is ; and so on. The surprising thing is that the sum of the internal angles of a polygon is not arbitrary, even though we can distort the figure in many ways.

Now we describe the second concept that we need. This is the similarity of two triangles.

Definition: If two triangles have the same set of angles, the two triangles are said to be **similar**.

Since the sum of the internal angles of a triangle is , if two angles of a triangle is equal to two angles of another triangle, then the triangles are similar. Basically two triangles are similar if they have the same "shape".

The fundamental property of similar triangles is that their **corresponding sides are proportional.** That is, in the figure below, the two triangles, and , are similar and is 4 times .

Then, and .

Let us see a consequence of these properties.

If in a triangle ABC, a line through the midpoint M of the side AC is drawn parallel to the side AB, then the line cuts the other side into two equal parts (That is, the line intersects the other side at the midpoint N of BC).

This is so because by the property of parallel lines, and , and so is similar to . Since , , and so N is the midpoint of the side BC. We also have .

14. In the above figure, draw a line through N parallel to the side AC, intersecting AB in O.

(a) If inches and the height inches, compute the areas of the four triangles, , , , and .

(b) If and , find the areas of the four triangles, , , , and .

15. Shown in the figure below is a parallelogram ABCD. Points E, F, G, and H are the midpoints of the sides of the parallelogram.

12.

(a) If the base of the parallelogram ABCD is 20 feet and the height is 8 feet, compute the areas of the four triangles at the "corner", namely, , , , and . (The height is not shown. Draw in the height yourself at an appropriate place or places.)

(b) Find the area of the inner figure EFGH.

(c) If the base of the parallelogram ABCD is b and the height is h, compute the areas of the four triangles, , , , and , and the area of the inner figure EFGH

6.2 Pythagorean Theorem

Pythagorean Theorem says that in a right triangle, the sum of the squares of the "legs" is equal to the square of the hypotenuse. (By "legs" of a right triangle, we mean the sides adjacent to the right angle, and of course the hypotenuse is the side opposite to the right angle.)

Pythagorean Theorem can be stated very simply using a figure. In the right triangle ABC with the right angle at B, if a and b are the legs and c is the hypotenuse, then

Pythagorean Theorem is one of the fundamental theorems in mathematics because so many things are based on it. Mathematicians all over the world found many proofs of the theorem. We will give one that is based on the material in the last section.

Given a right triangle with legs a and b, we construct a square by extending the legs so that one side of the square is , as shown in the figure.

We compute the area of the inner figure in two ways and equate the results:

(i) We compute the area A of the inner figure by subtracting the sum of the areas of the four triangles from the area of the (larger) square:

=

=

(ii) We observe that the inner figure is in fact a square. (This needs a justification.) So,

Equating these two results, we have

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Let us look at an example. Suppose we have a 120 yards by 80 yards rectangular playing field , as shown in the figure.

Experience tells us that it is a lot shorter to go from A to C directly than going from A to B and then to C. Pythagorean Theorem tells us the precise difference. Let us denote the distance from A to C by d. Then,

So,

yards

Therefore, if we go directly from A to C, the distance is about 144 yards as compared to 200 yards, which is the distance from A to B and then to C.

Exercises 6.2

1. Shown in the figure is a rectangle ABCD with and Compute the length of the diagonal AC.

2. A square is inscribed in a circle of radius 10 feet as shown in the figure below.

(a) Find the length of one side of the square.

(b) Find the area of one of the "crescents", the regions inside the circle but outside of the square.

3. Find the area of the isosceles triangle ABC if and

4. Find the area of the equilateral triangle if one side of the triangle is 20 feet. (Draw the figure yourself.)

5. The description of a television set says that the size of the screen is 20 inches, meaning that the length of the diagonal of the screen is 20 inches. Assuming that the TV screen is a square, compute the length of one side of the TV screen.

6. Find the area of the figure given below.

7. Find the area of the regular hexagon inscribed in a circle of radius 10 feet.

8. In the figure below, the equilateral triangle is topped by a semicircle. Find the total area of the figure.

9. In the figure below, the rectangle is topped by an **isosceles right triangle**. Find the area of the pentagon ABCDE.

10. Shown below is the right triangle ABC with and

(a) Compute the side .

(b) Let be the area of the equilateral triangle with as one side, be the area of the equilateral triangle with as one side, and be the area of the equilateral triangle with as one side. Compute , , , and , and compare and .

(c) Let be the area of the semicircle with as the diameter, be the area of the semicircle with as the diameter, and be the area of the semicircle with as the diameter. Compute , , , , and compare and .

(d) Let be the area of the **isosceles right** triangle with as the hypotenuse, be the area of the isosceles right triangle with the hypotenuse, and be the area of the isosceles right triangle with as the hypotenuse. Compute , , , , and compare and .