

## Operations with Algebraic Expressions

### 4.1 Expansion of products and simplification

We sometimes encounter situations in which we have to expand (or multiply out) the products like  $(a+b)(c+d)$ ,  $(a+b)(c+d+e)$ , etc. We will see how we can do this. It involves applications of the distributive property that we studied in Chapter 1. We recall that the distributive property says

for any numbers  $a$ ,  $b$ , and  $c$ , and

for any numbers  $a$ ,  $b$ , and  $c$ .

We will look at how we can expand the three products given above. In expanding the product  $(a+b)(c+d)$ , we first regard the first factor as one number and distribute over the terms of the second factor as follows:

$$=$$

$$=$$

$$=$$

You observe that the only thing new is the first step. So the expansion of products is very straightforward. Now let us look at the second example, again regarding the first factor as one number:

$$=$$

$$=$$

$$=$$

We note that we can obtain the second line by taking the first term of the second factor and distributing it over the terms of the first factor, then taking the second term of the second factor and distributing it over the terms of the first factor, and finally, taking the third term of the second factor and distributing it over the terms of the first factor. (This description may be confusing. The instructor can explain a lot more clearly on the board than we can describe in print.)

We will use the above observation to expand the third product:

$$=$$

2.

=

As a final example, we show how the necessity for expanding products comes about.

**Example:** Simplify the expression .

Solution: =

=

=

=

What this result says is that no matter what the numbers a and b are, the difference of the square of the sum and the square of the difference of a and b is always equal to 4 times the product of a and b. For example, if 7 and 4, then 11 and 3, so that

$$\begin{aligned} &= 11^2 - 3^2 = 121 - 9 = 112 \\ \text{while} &= 4(7)(4) = 112 \end{aligned}$$

In fact, we can check our algebraic computations by evaluating the original expression and the final expression for arbitrary values of the variables as in the case given above.

**Exercises 4.1**

1. Expand the following products and simplify:

(a)  $(x + 2)(x + 3)$  (b)  $(x + 4)(x + 5)$

(c)  $(x + 6)(x + 7)$  (d)  $(x + 8)(x + 9)$

(e)  $(x + 10)(x + 11)$  (f)  $(x + 12)(x + 13)$

(g)  $(x + 14)(x + 15)$  (h)  $(x + 16)(x + 17)$

2. From the results of the problems given above, what observations do you make?

3. Expand the following products and simplify:

(a)  $(x + 2)(x + 3)$  (b)  $(x + 4)(x + 5)$

(c)  $(x + 6)(x + 7)$  (d)  $(x + 8)(x + 9)$

(e)  $(x + 10)(x + 11)$  (f)  $(x + 12)(x + 13)$

(g)  $(x + 14)(x + 15)$

4. Simplify the following expressions:

(a)  $(x + 2)(x + 3)$  (b)  $(x + 4)(x + 5)$

(c)  $(x + 6)(x + 7)$  (d)  $(x + 8)(x + 9)$

(e)  $(x + 10)(x + 11)$  (f)  $(x + 12)(x + 13)$

5. Expand the products and make observations:

(a)  $(x + 2)(x + 3)$  (b)  $(x + 4)(x + 5)$

(c)  $(x + 6)(x + 7)$  (d)  $(x + 8)(x + 9)$

(e)  $(x + 10)(x + 11)$

6. From the results of Problem 5, what can you guess the result of expanding the product will be? From this result, can you guess the formula for the sum ? From this, can you guess the formula for the sum ?

4.

## 4.2 Factorization

When we apply the distributive property one way, we call the process expansion of product, and when we apply the same property in the reverse direction, we call the process factorization. So the concept is nothing new. For example, if we want to find the sum

we can add the numbers as written or we can rewrite the sum as

and then we find the sum of the numbers from 1 to 9, which we know to be 45, and multiply by 100. In this way we get 4500. In this case we say that we "**factored out the common factor** of 100" from the sum before adding the numbers. Let us now look at algebraic cases.

**Example :** Factor out the "largest" common factor from each of the following expressions:

(a)

(b)

(c)

(d)

(e)

(f)

Solution: (a) =

(b) =

(c)

(d)

(e)

(f)



6.

expanded. Your instructor can explain far more clearly on the board than we can here, so listen to your instructor's explanations carefully.

**Exercises 4.2**

1. Factor the following expressions:

- (a) (b)
- (c) (d)
- (e) (f)
- (g) (h)
- (i) (j)
- (k) (l)
- (m) (n)
- (o) (p)

2. Factor the following expressions:

- (a) (b)
- (c) (d)
- (e) (f)
- (g) (h)
- (i) (j)
- (k) (l)
- (m) (n)

3. Factor the following expressions as completely as possible. (You may have to factor out a common factor first.)

- (a) (b)
- (c) (d)
- (e) (f)

8.

- |     |     |
|-----|-----|
| (g) | (h) |
| (i) | (j) |
| (k) | (l) |
| (m) | (n) |
| (o) | (p) |
| (q) | (r) |
| (s) | (t) |
| (u) | (v) |
| (w) | (x) |
| (y) | (z) |

4. The following expressions, if factorable, can be factored by so-called "grouping". The idea is to get a binomial common factor, that is, a common factor consisting of two terms, like in the following expression:

.

Here is a common factor, and so the above expression can be factored as

.

It is instructive to expand the product and obtain the expression without any parentheses. We obtain

Notice the pattern in the coefficients (without the signs):

6, 3, 10, 5

The second number is half of the first and the fourth number is half of the third. For an expression consisting of 4 terms to be factorable by "grouping", the coefficients must have a similar pattern when the expression is written in a certain definite form,

like the descending order of the exponents of the variable.  
Examples of the patterns of the coefficients (without the signs)  
are:

1, 1, 1, 1  
1, 2, 1, 2  
2, 3, 2, 3,  
2, 6, 4, 12  
1, 5, 3, 15

For example, the expression is likely to be factored by grouping  
while the expression cannot be factored by the grouping  
method. To factor the first expression, we do as follows:

=

=

Factor the following expressions:

(a)

(b)

(c)

(d)

(e)

(f)

10.

### 4.3 Reduction of fractions

We recall that when we had a complicated fraction like  $\frac{12}{18}$ , we tried to reduce it to a simpler fraction by dividing the numerator and the denominator by a same number. In this case the numerator and the denominator are both even numbers, so that the fraction is clearly reducible. So, we divide the numerator and the denominator by 2 and get the fraction  $\frac{6}{9}$ , which is clearly reducible again. In arithmetic we continued the process until we could no longer reduce the fraction. The trouble with this method is that it does not carry over to algebra. The method that carries over to algebra is to decompose the numerator and the denominator into prime factors and then see what factors can be "canceled out". For example, for the fraction given above, we have

=

So, in order to reduce an algebraic fraction (if the fraction is reducible), we factor the numerator and the denominator completely and see which factor can be cancelled out.

**Example:** Simplify (or reduce) the following expressions:

(a)

(b)

(c)

(d)

Solution: (a)

(b)

(c)

(d)

**Exercises 4.3**

1. Simplify the following expressions:

- |     |     |
|-----|-----|
| (a) | (b) |
| (c) | (d) |
| (e) | (f) |
| (g) | (h) |
| (i) | (j) |
| (k) | (l) |
| (m) | (n) |
| (o) | (p) |
| (q) | (r) |
| (s) | (t) |
| (u) |     |

12.

## 4.4 Multiplication of fractions

We recall from arithmetic that the numerator of a product of fractions is the product of the numerators and the denominator of the product is the product of the denominators. In symbol, we can state this principle succinctly as

.

This principle of course carries over to algebra since the letters we use stand for unspecified numbers. After taking the product, we should examine whether the resulting fraction can be reduced or not, and if it is reducible, we should reduce it to the simplest form.

We recall that division by a fraction is multiplication by its reciprocal, and so we should convert division into multiplication immediately if the divisor is a simple fraction.

**Example:** Simplify the following fractions:

(a)

(b)

(c)

(d)

(e)

Solution: (a)

(b)

=

=

(c)

=

(d)

=

=

(e)

=

=

14.

### **Exercises 4.4**

Simplify the following expressions as far as possible:

(a)

(b)

(c)

(d)

(e)

(f)

(g)

(h)

(i)

(j)

(k)

(l)

(m)

(n)

(o)

(p)

(r)

(t)

(u)

(v)

## 4.5 Addition of fractions

Before we go into addition of algebraic fractions, let us review the absolute fundamentals.

By the fraction  $\frac{3}{4}$ , we mean three one-fourth, and so when we add fractions like  $\frac{1}{4}$  and  $\frac{2}{4}$ , we write them as  $\frac{1}{4}$  and  $\frac{2}{4}$  and apply the distributive property as follows:

This explains why in adding fractions having the same denominator, we only add the numerators. So, here is the principle of adding fractions when they have the same denominator:

We recall that "subtraction" of a number is defined as the addition of its negative, so that the above principle covers "subtraction" as well. It is important to note, however, that in expressions like the following

the minus sign in front of the second fraction applies to the whole numerator of the second fraction, so that it is a good idea to rewrite the expression as

before carrying out the computations.

We emphasize that in reducing a sum of fractions to a single fraction, the fractions to be summed must have a "common denominator". Your instructor can explain much better than we can here when the fractions have different denominators, and so we merely give examples below.

**Example 1:** Simplify the following expressions:

(a)

(b)

(c)

16.

(d)

(e)

(f)

(g)

Solution: (a)

(b)

=

=

=

=

(c)

=

=

(d)

=

=

=

(e)

=

=

(f)

=

=

(g)

=

=

If you understand these examples, you are ready to do the problems in Problem 1 in the exercises 4.5.

To do the rest of the problems you will have to use the least common denominator (LCD) or the smallest denominator that you can use. Your instructor will explain how to find it. We will just give examples:

**Example 2:** Simplify the following expressions as far as possible:

(a)

(b)

(c)

Solution: (a)

=

=

=

(b)

=

=

=

(c)

=

=

18.

=

**Example 2:** Simplify the following expression as far as possible:

**Solution:** We note that each factor is a simple expression. So, simplifying each expression separately, we get

and

So, we obtain

=

=

**Exercises 4.5**

1. Simplify the following expressions as far as possible:

(a)

(b)

(c)

(d)

(e)

(f)

(g)

(h)

(i)

(j)

(k)

(l)

(m)

(n)

(o)

(p)

(q)

(r)

(s)

(t)

(u)

(v)

(w)

(x)

2. Simplify the following expressions as far as possible:

(a)

(b)

(c)

(d)

20.

(e)

(f)

(g)

(h)

(i)

(j)

(k)

(l)

(m)

(n)

(o)

(p)

(q)

(r)

(s)

(t)

(u)

3. Simplify the following expressions:

(a)

(b)

(c)

(d)

(e)

(f)

(g)

(h)

(i)