Properties of Numbers

1.1 Commutative and associative properties of addition

When we do a lot of additions of numbers, we are bound to notice certain patterns. For example, we notice that in adding three or more numbers, it does not matter how we add the numbers. That is, if we want to add the three numbers, 7, 15, and 23, it does not matter whether we add 7 and 15 first (which gives 22) and then add 23, or 15 and 23 first (which gives 38) and then add the result to 7, or add 7 and 23 first (which gives 30) and then add 15. In any case, we obtain 45. In symbols the above computational procedures can be stated as

$$(7 + 15) + 23 = 22 + 23 = 45$$

$$(15 + 23) + 7 = 38 + 7 = 45$$

$$(7 + 23) + 15 = 30 + 15 = 45$$

The reason why these computational procedures give the same number becomes clear when we consider how such a sum arises. Suppose that three persons owe you money in the amounts of $7, $15, and $23. We know that the total amount you will collect does not depend on the order in which they pay you.

What the example indicates is that in summing a series of numbers, we can rearrange and regroup the numbers. These two properties --- our being able to rearrange and regroup --- are very useful and so they are given names. The first property, that is, our being able to rearrange the numbers, is called the commutative property (of addition), and the second property is called the associative property (of addition). To show their usefulness, consider the problem of finding the sum

$$7 + 9 + 5 + 3 + 5 + 8 + 7 + 8 + 3$$

Knowing that a combination of numbers like 5, 7, and 8 add up to 20, we rearrange and regroup the numbers and write the sum as

$$(7 + 5 + 8) + (5 + 8 + 7) + (9 + 3 + 3)$$

$$= 20 + 20 + 15$$

$$= 55$$

Of course, we can rearrange and regroup in other ways. In fact, it is a good practice to get the sum in several ways. For example, we can regroup as follows:

$$7 + 9 + 5 + 3 + 5 + 8 + 7 + 8 + 3$$

$$= (7 + 3) + (5 + 5) + (7 + 3) + (9 + 8 + 8)$$

$$= 10 + 10 + 10 + 25$$

$$= 55$$

Or we can regroup as

7 + 9 + 5 + 3 + 5 + 8 + 7 + 8 + 3

= (7 + 3) + (9 + 8 + 3) + (5 + 8 + 7) + (5)

= 10 + 20 + 20 + 5

= 55

These properties not only hold for the whole numbers but for fractions and negative numbers as well. We recall that **"subtraction"** is addition of a negative number. For example, the expression 5 - 8 + 7 is to be interpreted as 5 + (-8) + 7. Hence, **when we speak of "sum of numbers", the numbers may include negative numbers.**

Example 1: Find the sum

70 - 5 + 70 - 4 + 70 - 3 + 70 - 2 + 70 - 1 + 70 + 70 + 1 + 70 + 2 + 70 + 3 + 70 + 4 + 70 + 5.

Solution: By counting, we see that there are eleven 70's, and so we can write

The sum = 11(70) + (-5 - 4 - 3 - 2 - 1 + 1 + 2 + 3 + 4 + 5)

= 11(70) + 0 = 770

Example 2: Find the sum .

Solution: = (5 + 4 + 7 + 6) +

= 22 + =

Exercises 1.1

1. Compute the following sums by making use of the properties of numbers:

(a) 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 =

(b) 1 + 9 + 2 + 8 + 3 + 7 + 4 + 6 + 5 =

(c) 1 + 3 + 5 + 7 + 9 + 9 + 7 + 5 + 3 + 1 =

(d) 2 + 4 + 6 + 8 + 8 + 6 + 4 + 2 =

(e) (4 + 5 + 6) + (5 + 6 + 7) + (6 + 7 + 8) + (8 + 9 + 10) =

(f) 10 - 1 + 10 + 2 + 10 - 3 + 10 + 4 + 10 - 5 + 10 + 6 =

(g) 10 + 2 + 20 + 30 - 2 + 40 - 3 + 50 + 3 + 60 - 5 =

(h) 50 - 3 + 50 - 2 + 50 + 4 + 50 - 1 + 50 - 2 + 50 - 3 + 50 - 4 =

(i) 35 + 36 + 37 + 38 + 39 + 40 + 41 + 42 + 43 + 44 + 45 =

2. Adding two digit numbers is essentially the same as adding one digit numbers. By the number 57, for example, we mean 50 + 7. So, for example, in finding the sum

$$24 + 27 + 35 + 48 + 59,$$

we write:

$$24 + 27 + 35 + 48 + 59$$

$$= (20 + 20 + 30 + 40 + 50) + (4 + 7 + 5 + 8 + 9)$$

$$= 160 + 33$$

$$= 193$$

Find the following sums:

(a) 10 + 11 + 12 + 13 + 14 + 15 + 16 + 17 + 18 + 19

(b) 30 + 32 + 34 + 36 + 38 + 40 + 42 + 44 + 46 + 48

(c) 50 + 51 + 52 + 53 + 54 + 55 + 56 + 57 + 58 + 59 + 60 + 61 + 62

+ 63 + 64 + 65 + 66 + 67 + 68 + 69

(d) 12 + 23 + 34 + 45 + 56 + 67 + 78 + 89 + 91

(e) 98 + 87 + 76 + 65 + 54 + 43 + 32 + 21 + 19

(f) 1 + 3 + 5 + 7 + 9 + 11 + 13 + 15 + 17 + 19 + 21 + 23 + 25 + 27

+ 29 + 31 + 33 + 35 + 37 + 39

(g) 11 + 12 + 13 + 14 + 15 + 16 + 17 + 18 + 19 + 91 + 81 + 71 + 61

+ 51 + 41 + 31 + 21 + 11

3. Without actually computing, can you tell which of the following two sums is larger?

Sum1 = 11 + 22 + 33 + 44 + 55 + 66 + 77 + 88 + 99

Sum2 = 91 + 82 + 73 + 64 + 55 + 46 + 37 + 28 + 19

4. What is the sum of all the whole numbers from 10 to 60?

5. What is the sum of all the odd numbers from 11 to 59?

6. What is the sum of all the even numbers from 12 to 60?

7. Suppose it is known that the sum of all the integers from 1 through 100 is 5050. What is the sum of all the integers from 101 through 200?

8. Find the following sum:

(a)

(b)

(c)

(d)

1.2 Commutative and associative properties of multiplication

We have similar properties for multiplication. That is, in multiplying three or more numbers, we can rearrange and regroup the numbers. Thus, for example, in computing the product

$$4(7)(25)$$

we can multiply 4 and 7 (which gives 28) and then multiply by 25 to get the result 700. Or we can multiply 4 and 25 (which gives 100) and then multiply by 7 to get 700. In symbols, the above computational procedures can be expressed as

$$4(7)(25) = 28(25) = 700$$

$$(4 \cdot 25)(7) = 100(7) = 700$$

Because of the importance of these properties, they are given names. The property that two numbers can be multiplied in any order, that is, for example, 7 times 25 or 25 times 7, is called the commutative property (of multiplication), and the property that we can group the numbers in a product, such as $(4 \cdot 7)(25)$ or $4(7 \cdot 25)$, is called the associative property (of multiplication). Again these properties hold for fractions as well as for negative numbers. Moreover, it is useful to know some combination of numbers that multiply to 10 or 100, such as 2 and 5, and 4 and 25, and the fractional representations for some common decimal numbers like the following:

From these we can obtain others like and Thus, for example, instead of multiplying by 25, we can divide by 4 and multiply by 100.

We note that **"division by a number"** is defined as the **"multiplication by its reciprocal"**. Thus, for example, if we have , we write it as . We do this so that we can rearrange the numbers in computing a product of numbers.

We now give some examples of the uses of these equivalent forms.

Example 1: Compute .

Solution: $= 4(4)(4)(75)(75)(75)$

$= 4(4)(4)(25)(25)(25)(3)(3)(3)$

$= (100)(100)(100)(27)$

$= 27{,}000{,}000$

Example 2: Compute 0.75(28).

Solution: We make use of the fact that . Then,

So, $0.75(28) = 21$.

Exercises 1.2

1. Compute the following products by making use of the properties of numbers:

(a) 2(7)(5)(9) =

(b) 4(9)(15) =

(c) 3(4)(25)(27) =

(d) 2(2)(2)(2)(5)(5)(5)(5) =

(e) 4(4)(4)(25)(25)(25)(25) =

(f) 25(3)(4)(9) =

(g) 4(7)(5)(5)(5) =

(h) 25(84) =

(i) 75(4)(6) =

(j). 0.5(64)

(k). 0.25(144)

(l). 0.75(48)

(m). 2.5(36)

(n) 25(72)

(o). 75(84)

2. If Jane has 456 bags of candies each bag containing 78 candies, and Tom has 78 bags of candies each bag containing 456 candies, who has more candies?

3. Compute the following:

(a) =

(b) =

(c) =

(d) =

(e) =

(f) =

1.3 Distributive property

We now come to the third property, the distributive property (or the distributive law, as it is often called). Again we will illustrate the property using a problem of counting. Suppose we want to count the squares in the following figure.

We can count the squares in various ways:

1. We can regard the collection to be consisting of 15 columns, each column consisting of 5 squares. Therefore,

$$\text{The total number of squares} = 15(5) = 75$$

2. We can divide the collection into two parts, one part consisting of 8 columns and the other part consisting of 7 columns, each column consisting of 5 squares. So,

$$\begin{aligned}\text{The total number of squares} &= 8(5) + 7(5)\\ &= 40 + 35\\ &= 75\end{aligned}$$

3. We can divide the collection into three parts, one part consisting of 4 columns, the second part consisting of 6 columns, and the third part consisting of 5 columns. Then,

$$\begin{aligned}\text{The total number of squares} &= 4(5) + 6(5) + 5(5)\\ &= 20 + 30 + 25\\ &= 75\end{aligned}$$

Thus, we have

$$(8 + 7)(5) = 8(5) + 7(5)$$

and

$$(4 + 6 + 5)(5) = 4(5) + 6(5) + 5(5)$$

The distributive law is usually stated in the following way:

For any numbers a, b, and c,

$$a(b + c) = ab + ac$$

and

$$(b + c)a = ba + ca.$$

The distributive property is extremely useful. We now give examples of the use of the distributive property.

Example 1: Find the estimate of the product 7(39) first, and then obtain the exact value by applying the distributive property, and observe the error being committed in obtaining the estimate.

Solution: Since 39 is very close to 40, we reason that 7(39) must be close to 7(40). Therefore,

$$7(39) \approx 7(40) = 280$$

So, 7(39) is approximately equal to 280. We now compute the exact value by applying the distributive property:

$$\begin{aligned} 7(39) &= 7(40 + (-1)) \\ &= 7(40) + 7(-1) \\ &= 280 - 7 \\ &= 273 \end{aligned}$$

Thus the exact value of the product 7(39) is 273. By approximating 39 by 40 in the product 7(39), we committed the error of 7.

Example 2: Compute 76(9.5).

Solution: Since 9.5 is close to 10, we estimate that the product must be close to 76(10) or 760. We now compute the exact value:

$$76(9.5) = 76(10 - 0.5)$$

$$=$$

14.

$=$

$=$

$= 722$

So, $76(9.5) = 722$.

Example 3: At a restaurant, the menu says that a complete dinner costs $19.75 per person. If you have a party of 8 persons, what will be the cost of the dinner for your party?

Solution: The cost of the dinner = 8(19.75) dollars.

Let us first make an estimate of the cost. Since $19.75 is close to $20, the cost must be close to 8(20) or 160 dollars. Now we compute the cost exactly, making use of the fact that
19.75 = 20 - 0.25 and :

=

=

= 158

So, the cost of the dinner for your party will be $158.

Notice that in every one of the cases in which we made estimates in the above examples, we made use of the estimate in the computation of the exact value. So, making an estimate before going into the exact computation is not a waste of time. We highly recommend making an estimate before plunging into the exact computation.

Exercises 1.3

1. In each case, obtain an estimate of the product, and then compute the exact value using the distributive property:

(a) 25(39) =

(b) 24(48) =

(c) 25(49) =

(d) 76(48) =

(e) 74(36) =

2. Compute the following:

(a). 4.95(16)

(b). 6.25(32)

(c). 4.75(12)

(d). 9.5(28)

(e). 0.15(68)

3. In each of the following problems, make a quick estimate of the quantity to be computed, and then find the exact value of the quantity making use of the properties of numbers:

(a) If the price of Aku is $5.95 per pound, what is the price of 7 pounds of Aku?

(b) The price of watermelon is 29¢ a pound. What is the price of an 18-pound watermelon?

(c) When you ask a person from the Mainland how far a certain place is from his home, he will tell you something like, "It is a two-and-a-half-hour drive." Now supposing that he can drive at 50 miles per hour, what distance in miles is he talking about?

(d) The price tag on an aloha shirt says $29.95, but the advertisement says that the price of every item is reduced by 30%. What is the sale price of the aloha shirt?

(e) The price of turkey is 59¢ per pound. What is the price of a 25-pound turkey?

(f) At a restaurant the price of a dinner is listed as $17.50 per person. If you have a party of 16 persons, what will be the cost of dinner for you party? If you want to leave a 15% tip, what will be the tip?

1.4 Applications to algebra

The reason why you have to be thoroughly familiar with the properties of numbers is that they are used in algebra without being mentioned. It is assumed that you are thoroughly familiar with those properties so that you can recognize whenever they are used. We now show how those properties are used in simplifying algebraic expressions. To simplify an algebraic expression means to write expression in a compact form or easily computable form. Before we illustrate what we mean, let us recall the following facts

* Whenever we use letters like x, y, z, a, b, c, u, w, etc., we mean that the letters stand for unspecified numbers.

* Exponential notation:
By , we mean the product of 2 x's. That is, .
By , we mean .
By , we mean .
And so on.
For example, the product can be written as (and they are equal to 32).

* When we see expressions like and , it is useful to regard them as and , respectively.

Example 1: Simplify the expression .

Solution: We rearrange and regroup the numbers in the product as follows:

=

=

Example 2: Simplify the expression .

Solution: =

=

=

Before we illustrate the use of the distributive property, we recall that by the expression , we mean

Example 3: Simplify the expression .

Solution: =

=

=

=

Here we made use of the fact .

Example 4: Simplify the expression .

Solution: =

=

=

=

Here we rearranged the terms and combined "similar terms".

At first it is a good practice to write down all the steps as in the above examples so that you can visualize the steps that you will be omitting later on. However, it is tedious to write down all the steps and writing down all the steps may lead to miscopying mistakes. So, we usually omit quite a few of the steps from the above procedures to increase accuracy. What is important is that we can carry out the simplification accurately.

Exercises 1.4

Simplify the following expressions:

1.

2.

3.

4.

5.

6.

7.

8.

9.

10.

11.

12.

13.

14.

15.

16.

17.

18.