5.5 Deviation I

Let $a_1,\ a_2,\ a_3,\ .\ .\ .\ ,a_n$ be n numbers, and let

$$\delta(x) \ = \ |x-a_1| + |x-a_2| + |x-a_3| + \ .\ .\ . + |x-a_n|$$

$\delta(x)$ is the sum of absolute values of the differences of x from the numbers $a_1,\ a_2,\ a_3,\ .\ .\ .\ ,a_n,$ and so is positive. One interpretation of $\delta(x)$ is that it is the deviation of x from the numbers $a_1,\ a_2,\ a_3,\ .\ .\ .\ ,a_n$. Usually these numbers are close together, for this study we separate them out so that we can see the behavior of $\delta(x)$ better.

Recall that by the absolute value of a number u, we mean the magnitude of u. Thus, for example, |5| = 5 and |-5| = 5.

We begin with a very simple function.

Example 1: Let $f(x) \ = \ |x-3| + |x-8|$

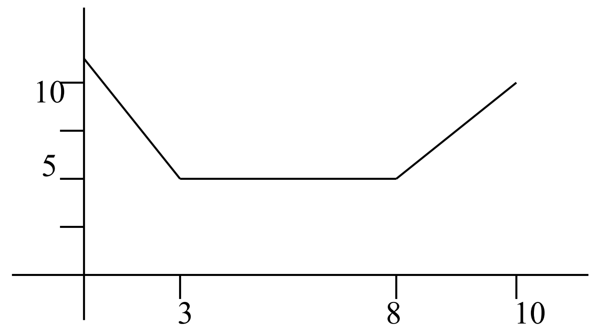
Let us make some tables of values of this function and see what we observe.

x	f(x)	x	f(x)	x	f(x)
0	11	3	5	8	5
1	9	4	5	9	7
2	7	5	5	10	9
3	5	6	5		
		7	5		
		8	5		

We see that f(x) is a piece-wise linear function, and another description of f(x) is

$$x \le 3, \quad f(x) = -2(x-3) + 5$$
$$3 \le x \le 8, \quad f(x) = 5$$
$$x \ge 8, \quad f(x) = 2(x-8) + 5$$

The graph of f(x) looks like this:



We see that the smallest value of f(x) is 5, and f(x) is smallest at any value of x between 3 and 8.

Let us look at a slightly more complicated function.

Example 2: Let $g(x) = |x-2| + |x-5| + |x-10|$

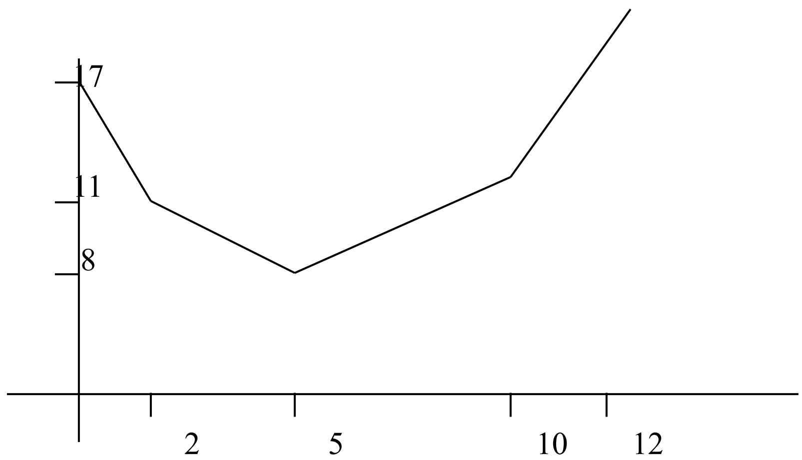
Again let us make tables of values of g(x):

x	g(x)	x	g(x)	x	g(x)	x	g(x)
0	17	2	11	5	8	10	13
1	14	3	10	6	9	11	16
2	11	4	9	7	10	12	19
		5	8				
				10	13		

Again g(x) is a piece-wise linear function, and its another description is

$$x \le 2, \quad g(x) = -3x + 17$$
$$2 \le x \le 5, \quad g(x) = -1(X-2) + 11$$
$$5 \le x \le 10, \quad g(x) = 1(x-5) + 8$$
$$x \ge 10, \quad g(x) = 3(x-10) + 13$$

The graph of g(x) looks like the following:



We observe that the smallest value of g(x) is 8, and the smallest value of g(x) is attained at x = 5. Note that 5 is the midvalue of th three numbers, 2, 5, and 10.

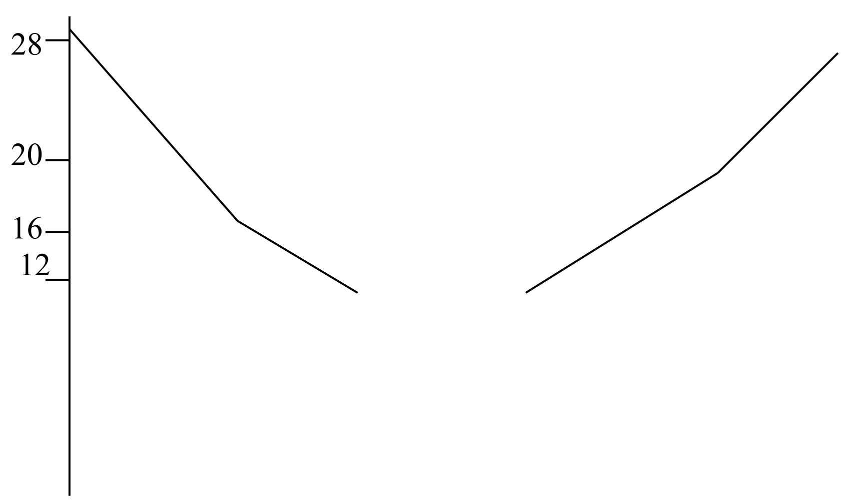
Let us look at one more example.

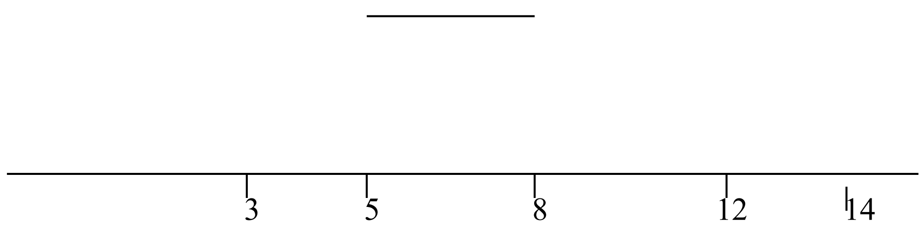
Example 3: Let $h(x) = |x-3| + |x-5| + |x-8| + |x-12|$

Let us make some tables again:

x	h(x)	x	h(x)	x	h(x)	x	h(x)	x	h(x)
0	28	3	16	5	12	8	12	12	20
1	24	4	14	6	12	9	14	13	24
2	20	5	12	7	12	10	16	14	28
3	16			8	12	11	18		
						12	20		

When we sketch the graph of h(x), we get something like the following:





We notice the following:

(a) The break in the lines occur at the numbers 3, 5, 8, and 12.

(b) h(x) decreases and then increases as x varies from 3 to 12.

(c) The smallest value of h(x) occurs when x is any value between 5 and 8.

(d) The graph is essentially different when the number of the given numbers, $a_1, a_2, a_3, \ldots, a_n$, is even or odd.

When a collection of numbers is arranged in ascending (or descending) order, the **median** of the collection is defined to be the middle number if the number of the numbers is odd and the average of the middle two numbers if the number is even. For the collection $\{2, 5, 10\}$, the median is 5; for the collection $\{3, 5, 8, 12\}$, the median is $\frac{5+8}{2} = 6.5$. With this language, we have the following result:

The deviation function

$$\delta(x) = |x - a_1| + |x - a_2| + |x - a_3| + \ldots + |x - a_n|$$

is minimized by the median of the numbers $a_1, a_2, a_3, \ldots, a_n$.

Exercise 5.5

1. Let $f(x) = |x-3| + |x-5| + |x-8| + |x-9| + |x-12|$
   Make tables of values for f(x), sketch the graph, find the smallest value of f(x), and the value of x at which f(x) is smallest.

2. Let $g(x) = |x-3| + |x-5| + |x-8| + |x-9| + |x-12| + |x-15|$
   Make tables of values for g(x), sketch the graph, find the smallest value of g(x), and the value of x at which g(x) is smallest.

5.6 Deviation II and minimizing quadratic functions

Let $a_1,\ a_2,\ a_3,\ .\ .\ .\ ,a_n$ be n numbers, and let

$$\delta_2(x) \ = \ (x-a_1)^2+(x-a_2)^2+(x-a_3)^2+\ .\ .\ .+(x-a_n)^2$$

$\delta_2(x)$ is the sum of the squares of the differences of x from the numbers $a_1,\ a_2,\ a_3,\ .\ .\ .\ ,a_n,$ and so is positive. $\delta_2(x)$ is another deviation of x from the numbers $a_1,\ a_2,\ a_3,\ .\ .\ .\ ,a_n.$ We study this deviation in this section. We note that when we multiply out and collect similar terms, we can bring $\delta_2(x)$ to the form $ax^2+bx+c.$ So, we have to study functions of this form. As usual we begin with a very simple case.

Let $f(x)=5(x-7.5)^2+37.$

Fill in the following table:

x	$f(x)$
5.5	57
6.5	42
7.5	37
8.5	42
9.5	57

Since $f(x)$ is a sum of a square and a positive number for any x, $f(x)$ is positive for any value of x. Moreover, the smallest value of $f(x)$ occurs when the first term is 0, and so the smallest value of $f(x)$ is 37. The value of x that makes $f(x)$ smallest is, therefore, 7.5.

By just looking at the function $g(x)=75(x-127)^2+45,$ we see immediately that the smallest value of $g(x)$ is 45, and the value of x that makes $g(x)$ smallest is 12.7.

The point is, if we have a "**quadratic function**" $ax^2+bx+c,$ we can tell immediately what value of x minimizes the function if we bring the function to the form $a(x-d)^2+e.$ The process of bringing ax^2+bx+c to the form $a(x-d)^2+e$ is called "**completing the square"** and is a very important process. The process is based on a very simple formula

$$(x-a)^2=x^2-2ax+a^2$$

Let us look at some examples.

Example 1: Bring $x^2 - 8x$ to the form $a(x-d)^2 + e$.

Solution: $x^2 - 8x = (x-4)^2 - 16$

Because $(x-4)^2 = x^2 - 2(4)x + 4^2 = x^2 - 8x + 16$, so that to preserve the equality, we have to get rid of the 16 by subtracting it.

Example 2: Bring $x^2 - 15x$ to the form $a(x-d)^2 + e$.

Solution: $x^2 - 15x = \left(x - \frac{15}{2}\right)^2 - \frac{15^2}{4}$

If the coefficient of x^2 is not 1, we factor out the coefficient.

Example 3: Bring the expression $3x^2 - 7x$ to the form $a(x-d)^2 + e$.

Solution:

$$\begin{aligned} 3x^2 - 7x &= 3\left(x^2 - \frac{7}{3}x\right) \\ &= 3\left[\left(x - \frac{1}{2}\cdot\frac{7}{3}\right)^2 - \frac{49}{36}\right] \\ &= 3\left(x - \frac{7}{6}\right)^2 - 3\left(\frac{49}{36}\right) \\ &= 3\left(x - \frac{7}{6}\right)^2 - \frac{49}{12} \end{aligned}$$

Now let us get down to the real problem.

Example 4: Find the value of x that makes the function $f(x) = 5x^2 - 17x + 30$ smallest.

Solution:

$$\begin{aligned} f(x) &= 5\left[x^2 - \frac{17}{5}x\right] + 30 \\ &= 5\left[\left(x - \frac{1}{2}\cdot\frac{17}{5}\right)^2 - \frac{17^2}{100}\right] + 30 \\ &= 5\left(x - \frac{17}{10}\right)^2 - 5\left(\frac{17^2}{100}\right) + 30 \\ &= 5\left(x - \frac{17}{10}\right)^2 + 15.55 \end{aligned}$$

So, the value of x that makes f(x) smallest is $\frac{17}{10} = 1.7$.

Of course, our conclusion is bases on the last line, which we assume to be correct. Let us evaluate $f(x) = 5x^2 - 17x + 30$ at three values of x near 1.7 and at 1.7:

f(1.6) = 15.6
f(1.7) = 15.55
f(1.8) =15.6

Of the three values, f(1.7) is the smallest, and so we conclude that our computation is correct.

Example 5: Let $g(x) = (x-3)^2 + (x-5)^2 + (x-8)^2$.

Find the value of x that minimizes $g(x)$.

Solution: We first bring g(x) to the form $ax^2 + bx + c$ and then complete the square:

$$\begin{aligned} g(x) &= (x^2 - 6x + 9) + (x^2 - 10x + 25) + (x^2 - 16x + 64) \\ &= 3x^2 - 32x + 98 \\ &= 3\left[x^2 - \frac{32}{3}x\right] + 98 \\ &= 3\left[\left(x - \frac{16}{3}\right)^2 - \frac{16^2}{9}\right] + 98 \\ &= 3\left(x - \frac{16}{3}\right)^2 - 3\left(\frac{16^2}{9}\right) + 98 \\ &= 3\left(x - \frac{16}{3}\right)^2 + 12.6\overline{6} \end{aligned}$$

The value of x that minimizes g(x) is $\frac{16}{3} = 5.3\overline{3}$.

Let us check by computing g(x) at x = $\frac{16}{3}$ and both sides of $\frac{16}{3}$:

g(5) = 13
$g\left(\frac{16}{3}\right) = 12.6\overline{6}$
g(6) = 14

Since g(x) is smallest of the three values, we conclude that our answer is correct.

Exercises 5.6

1. Bring each of the following expressions to the form $a(x-d)^2+e$:

(a) x^2-16x

(b) x^2-19x

(c) $x^2-20x+150$

(d) $x^2-25x+160$

(e) $4x^2-23x+60$

(f) $5x^2-24x+40$

2. In each case, find the value of x that minimizes the function: (Check your answer in each case.)

(a) $f(x)=6x^2-25x+40$

(b) $g(x)=10x^2-34x+55$

(c) $h(x)=(x-3)^2+(x-5)^2+(x-9)^2$

(d) $k(x)=(x-3)^2+(x-5)^2+(x-8)^2+(x-9)^2$

(e) $L(x)=4x^2-23x+60$

(f) $p(x)=5x^2-24x+40$

3. In each case, find the value of x that minimizes the function: (Check your answer in each case.)

(a) $f(x)=6x^2-25x+40$

(b) $g(x)=10x^2-34x+55$

(c) $h(x)=(x-3)^2+(x-5)^2+(x-9)^2$

(d) $k(x)=(x-3)^2+(x-5)^2+(x-8)^2+(x-9)^2$