Chapter 4 Exponential Equations and Logarithms

4.1 Properties of exponents

We recall that we have the following properties of exponents if the exponents are positive integers:

(i) $a^m a^n = a^{m+n}$

(ii) $\frac{a^m}{a^n} = a^{m-n} \quad if \;\; m > n$

$\frac{a^m}{a^n} = \frac{1}{a^{n-m}} \quad if \;\; n > m$

(iii) $(a^m)^n = a^{m(n)}$

(iv) $(ab)^n = a^n b^n$

(v) $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$

It turns out that if we define negative and fractional exponents in the following manner, these properties hold even for fractional and negative exponents as well:

$a^0 = 1$ for any a not equal to 0.

$a^{-n} = \frac{1}{a^n}$ for n > 0. (Negative exponent means reciprocal.)

$a^{\frac{p}{q}} = (\sqrt[q]{a})^p$, where p and q are integers.

In order to preserve all properties of exponents listed above, however, we have to impose that a and b are **positive** numbers.

Examples:

$2^{-5} = \frac{1}{2^5} = \frac{1}{32}$

$10^{3} = \frac{1}{10^{3}} = 0.001$

$$3.54 \times 10^{4} = 3.54\left(\frac{1}{10^{4}}\right) = \frac{3.54}{10^{4}} = 0.00035$$

$$3^{\frac{2}{5}} = \left(\sqrt[5]{3}\right)^{2}$$

$$6^{0.25} = 6^{\frac{1}{4}} = \sqrt[4]{6}$$

These definitions enable us to compute numbers like $\left(\sqrt[5]{3}\right)^{2}$ and $\sqrt[4]{6}$ using the exponential key in the calculator:

$$\left(\sqrt[5]{3}\right)^{2} = 3^{\frac{2}{5}} = 3^{0.4} = 1.5518455$$

$$\sqrt[4]{6} = 6^{\frac{1}{4}} = 6^{0.25} = 1.565084$$

$$3.75^{\frac{1}{80}} = 3.75^{0.0125} = 1.016659$$

It should be noted that the equality signs just before the decimal representations are in the sense of **approximate equality**.

4.2 Exponential equations

The importance of fractional exponents and their properties lies in that the extension of the properties to fractional exponents (and eventually to all real exponents) enables us to solve exponential equations, that is, equations in which the unknowns occur in the exponents. For example, consider the following problem:

> If we invest $2,000 at the interest rate of 5.5% per annum compounded quarterly, how long will it take for the principal to become $3,000?

To answer the question, we have to solve the equation

for n, where n is in years. We divide both sides of the equation by 2000 and the write the equation in the form or

(1)

Now, there are two marvelous keys on a scientific calculator by which we can express any positive number as the power of 10 or as a power of another number e, which will be explained in the next section. The key that enables us to express any positive number as a power of 10 is the **log key**. For example, when we enter 7 and press the log key, we will get 0.84509804 (or a number very close to it). This means that

$$7 = 10^{0.8450980}$$

(The equality is in the sense of approximate equality.) We call the number 0.84509804 the **logarithm of 7 to the base 10** and denote it by the symbol

or simply when the base is understood to be 10.

Getting back to the problem of solving Equation (1), the idea is to express 1.0375 and 1.5 as power of 10. So, we enter 1.0375 and press the log key. We obtain 0.005930867 , so that

$$1.01375 = 10^{0.0059308}$$

Similarly, we express 1.5 as a power of 10 and obtain

$$1.5 = 10^{0.1760912}$$

Substituting these quantities for 1.01375 and 1.5 in Equation (1), we get

$$(10^{0.005930867})^{4n} = 10^{0.1760912}$$

Applying Property (iii) of exponents to the left-hand side of this equation, we have

$$10^{(0.005930867)(4n)} = 10^{0.1760912}$$

Since the bases on both sides of the equation are the same, the equation will be satisfied if we choose n so that the exponents are equal. Thus, we choose n so that

or

Hence, it takes a little more than 7.4 years (or 7 years and 5 months) for $2000 to become $3000 if invested at 5.5% per annum compounded quarterly.

Let us check our answer, taking n = 7.4227:

$$P_{7.4227} = 2000\left(1 + \frac{0.055}{4}\right)^{4(7.4227)} = 3000.00644$$

When we round off to the nearest cents, we get $3000.

Exercises 4.2

1. If $2000 is invested at 6.75% per annum compounded monthly, how long will it take for the principal to become $5000?

2. If $1000 is invested at 6% per annum compounded quarterly, how long will it take for thee principal to become:

(a) $2000?

(b) $3000?

(c) $4000?

3. If $1000 is invested at 8.5% per annum compounded monthly, how long will it take for the principal to become $5000?

4. If the current inflation rate is 5% per year, how long will it take for the price of goods to double, assuming that the inflation rate remains the same?

A much more interesting question is the following.

> If we deposit $100 a month (at the end of each month) into a savings account that pays 6% per annum compounded monthly and in which we already have $1000, how long will it take to accumulate the capital of $20,000?

To answer the question, we use the savings program formula

$$P_n = P_0\left(1+\frac{r}{12}\right)^n + \frac{M\left[\left(1+\frac{r}{12}\right)^n - 1\right]}{\frac{r}{12}}$$

We solve this equation for $\left(1+\frac{r}{12}\right)^n$ first. To facilitate writing, we let $u=\left(1+\frac{r}{12}\right)^n$ $and\ A=\frac{r}{12}$. Then, the above equation becomes

$$P_n = P_0 u + \frac{M(u-1)}{A}$$

When we solve this equation for u, we obtain (You should do it yourself.)

$$u=\frac{AP_n + M}{AP_0 + M}$$

Substituting back, we have

(2) $$\left(1+\frac{r}{12}\right)^n = \frac{\left(\frac{r}{12}\right)\cdot P_n + M}{\left(\frac{r}{12}\right)\cdot P_0 + M}$$

We recall that

P_0 = the initial amount in the savings account,
r = the interest rate per year (compounded monthly),
M = the monthly contribution (to be made at the end of each month),
P_n = the principal at the end of the **nth month**.

Now getting back to the question posed above, we have

$$\left(1+\frac{0.06}{12}\right)^n = \frac{\left(\frac{0.06}{12}\right)\cdot(20{,}000) + 10(}{\left(\frac{0.06}{12}\right)\cdot(1000) + 100}$$

Simplifying the equation, we have

$$(1.005)^n = 1.904761905$$

Expressing 1.005 and 1.904761905 in terms of 10, we have

$$(10^{0.002166062})^n = 10^{0.279840697} \quad \text{or}$$

$$10^{0.002166062(n)} = 10^{0.279840697}$$

Equating the exponents and solving for n, we get

$$n = \frac{0.279840697}{0.002166062} = 129.19331$$

Hence, it takes a little more than 129 months to accumulate the capital of $20,000.

Let us check using the savings program equation:

$$P_{129} = 1000\left(1+\frac{0.06}{12}\right)^{129} + (100)\left[\left(1+\frac{0.06}{12}\right)^{129} - 1\right]\cdot\left(\frac{12}{0.06}\right)$$

$$= 19961.45213$$

Therefore, at the end of the 129th month, we have $19,961.45, which is close to $20,000.

5. If you deposit $150 every month (at the end of each month) into a savings account in which you have $1000, how long will it take to accumulate the capital of $25,000 if the interest is:

   (a) 5.5% per annum compounded monthly?

   (b) 6% per annum compounded monthly?

   (c) 6.5% per compounded monthly?

6. If you begin a savings program by opening a savings account that pays 6% per annum compounded monthly, how long will it take to accumulate the capital of $100,000 if your monthly contribution is $400?

The following problem requires a formula similar to Equation (2) of this section. To get the formula, begin with the installment purchase formula

$$P_n = P_0\left(1+\frac{r}{12}\right)^n - \frac{M\left[\left(1+\frac{r}{12}\right)^n - 1\right]}{\frac{r}{12}}$$

and make similar substitutions as in the case given above and obtain the equation like Equation (2). But in this case, we have to find n so that $P_n = 0$. So, the equation to solve is

$$0 = P_0\left(1+\frac{r}{12}\right)^n - \frac{M\left[\left(1+\frac{r}{12}\right)^n - 1\right]}{\frac{r}{12}}$$

When we make the same substitutions, the above equation becomes

$$0 = P_0 u - \frac{M(u-1)}{A}$$

Solve this equation for u. You should get

$$u = \frac{M}{M - AP_0}$$

and substituting back, we get

$$\left(1+\frac{r}{12}\right)^n = \frac{M}{M - \left(\frac{r}{12}\right)P_0}$$

7. Suppose that we agree to buy a piece of land for $50,000, to be financed through the company selling the land at 10.5% per annum, and agree to make the payment of $500 per month.

(a) How much will we still owe at the end of the 10th year?

(b) What will be the amount of interest we will pay in the first 10 years?

(c) How long will it take us to pay off for the land?

(d) Express the answer for (c) in the form of so many payments of $500 each and the last payment of so many dollars.

4.3 Natural Logarithm

We mentioned in the preceding section that there is another key by which we can express any positive number as a power of another number. The key is the **natural log key**, denoted variously on scientific calculators as **lnx, ln, or LN**, and the number is e, whose approximate value is 2.718281828. One might ask why we use such a complicated number for a base when a very simple number like 10 is available. It just so happens that the number e is more natural than the number 10 in many respects. In any case, if we enter 7 and press the natural log key, we get 1.945910149. This means that

$$7 = e^{1.9459101}$$

The number 1.945910149 is called the logarithm of 7 to the base e or more commonly t**he natural logarithm of 7**, and is denoted by the symbol .

We want to illustrate the use of the natural logarithms by indicating how we could have solved the example in the preceding section. In that example, the problem boiled down to solving the exponential equation

Instead of expressing 1.01375 and 1.5 as powers of 10, we now express them as powers of e. So, we enter 1.01375 and press the natural log key, and obtain

$$1.01375 = e^{0.0136563}$$

Similarly, we obtain

$$1.5 = e^{0.4054651}$$

Substituting these expressions for 1.01375 and 1.5, respectively, in the above equation, we have

$$(e^{0.013656326})^{4n} = e^{0.4054651}$$

Applying the property of exponents on the left-hand side and equating the exponents, we obtain

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from which we get

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which is exactly the same number as the one we obtained in that example.

We want to indicate where the number e comes from and how such a number arises in practice. To see where the number e comes from, let us compute the following expressions:

We notice that the values of the expression seem to approach a certain number

as n becomes very large. In fact, the limiting number is defined to be e. In symbol, we write

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Let us see how the number e arises in practice.

Example Suppose $1,000 is deposited in a savings account that pays the interest of 6.5% per annum. Compute the principal at the end of the first year if the interest is compounded (i) monthly; (ii) daily; (iii) hourly.

Solution: (i)

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(ii)

(iii)

Notice that the last two figures do not differ very much. In fact, they differ by less than one cent. Now let us compute using 2.7182818 as the value of e. We will have 1067.159, which is very close to the last two figures. This is no coincidence. To see why, we put the expression in (ii) without the factor 1000 into a more illuminating form:

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=

The expression inside the brackets is of the form

with . Since n is large, the expression inside the brackets should be approximately equal to e. In fact, it is 2.7180398, which differs from e by about 0.0002. Thus,

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In general, when m is large, we have

and the approximation becomes better and better as m/r becomes larger and larger. Thus, when m is large, the compound interest formula can be approximated as

$$= P_0 e^{r(n)}$$

The formula

$$P_n = P_0 e^{r(n)}$$

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is known as the **continuous compound interest formula** (that is, the interest is compounded continuously). Notice how the formula simplified when we took the limiting case. It is a remarkable fact that when we increased the number of times the interest was compounded, the formula actually became simpler. This is not an isolated instance with such phenomena. It turns out that many problems do simplify when we consider the limiting cases if the limiting cases exist. The systematic study of those problems for which limiting cases exist forms the basis for the subject called calculus.

The above formula, moreover, highlights the fact that the principal at the end of the nth year is an exponential function of the interest rate r (as well as of n).

We could have solved all the problems in the preceding section using the natural logarithms instead of the logarithms to the base 10. The reason why we bring up the natural logarithms is that when you encounter logarithms in other courses like in physics and computer sciences and higher mathematics, the logarithms are most likely to be the natural logarithms.