Chapter 2 Mathematics of Finance

2.1 Compound interest

The kind of interest used by financial institutions today is compound interest. This means that the interest on a principal is computed and added to the principal after a specified period. Let us consider a specific example to get the concrete picture.

Example 1: Suppose we deposit $1000 in a savings account that pays 3.5% per year compounded quarterly. We want to know the principal at the end of the fifth year.

The interest being compounded quarterly means that the interest is computed at the end of the first quarter and added to the principal. On the second quarter, the interest is computed on the new amount. Let us use symbols to describe more precisely.

Let P_{01} = the principal at the end of the 1st quarter,
P_{02} = the principal at the end of the 2nd quarter,
P_{03} = the principal at the end of the 3rd quarter, and
P_{04} = the principal at the end of the 4th quarter, or at the end of the first year.

Then, $P_{01} = 1000 + 1000(0.035)\left(\frac{1}{4}\right) = 1008.75$ dollars

$P_{02} = 1008.75 + 11008.75\left(\frac{1}{4}\right) = 1017\left(\frac{1}{4}\right).576563$ dollars

$P_{03} = 1017.576563 + 10175.576563(0.035)\left(\frac{1}{4}\right) = 1026.480357$ dollars

$P_{04} = 1026.480357 + 1026.480357(0.035)\left(\frac{1}{4}\right) = 1035.462061$ dollars

We have retained all the figures because there is a formula by which we can compute the last figure without going through the intermediate results. The formula is

$$P_{04} = P_0\left(1 + \frac{r}{4}\right)^4$$

where P_0 is the original amount and r is the annual interest rate. In the above example, we have

$$P_{04} = 1000\left(1 + \frac{0.035}{4}\right)^4 = 1035.46206$$

which is exactly the same as the line in the above computation. We will show how the general formula is obtained in the next chapter. We merely give the result so that we can make use of the formula. The general formula related to this problem is this:

If P_n is the principal at the end of the nth **year**, then

$$P_n = P_0\left(1+\frac{r}{4}\right)^{4n}$$

To answer the question in the example,

$$P_5 = 1000\left(1+\frac{0.035}{4}\right)^{4(5)} = 1000\left(1+\frac{0.035}{4}\right)^{20} = 1190.339799$$

So, the amount we will have at the end of the 5th year is $1190.34.

Suppose we change the number of compounding in Example 1.

Example 2: Suppose we deposit $1000 in a savings account that pays 3.5% per year compounded monthly. What will be the principal at the end of the fifth year?

If we follow the steps in Example 1, we will go through 12 steps to get to the amount at the end of the 12th month or the first year. Here we merely give the formula. The formula looks like this:

$$P_1 = P_0\left(1+\frac{r}{12}\right)^{12}$$

and

$$P_n = P_0\left(1+\frac{r}{12}\right)^{12n}$$

So, to answer the question, we have to compute

$$P_5 = 1000\left(1+\frac{0.035}{12}\right)^{12(5)} = 1000\left(1+\frac{0.035}{12}\right)^{60} = 1190.942829$$

Therefore, the principal at the end of the 5th year will be $1190.94.

Some financial institutions compound interest daily. Let us see how the principal changes.

Example 3: Suppose we deposit $1000 in a savings account that pays 3.5% per year compounded daily. What will be the principal at the end of the fifth year?

We assume that 1 year consists of 365 days. We merely use the formula. The formula looks like this for this case: The principal P_n at the end of the nth **year** is

$$P_n = P_0\left[1+\frac{r}{365}\right]^{365n}$$

To answer our question, we compute

$$P_5 = 1000\left[1+\frac{0.035}{365}\right]^{365(5)} = 1000\left(1+\frac{0.035}{365}\right)^{1825} = 1191.236221$$

The principal at the end of the 5th year will be $1191.24 if the interests are compounded daily.

In general, we have the following **compound interest formula:**

If P_0 is invested at the rate r per year compounded m time a year, the principal P_n at the end of the nth **year** is given by

$$P_n = P_0\left[1+\frac{r}{m}\right]^{m(n)}$$

We stated above that it would not make any significant difference whether we regard 1 year to consist of 365 or 360 days. Let us justify our statement with Example 3. If we regard 1 year to consist of 360 days, we have

$$P_5 = 1000\left(1+\frac{0.035}{360}\right)^{360(5)} = 1191.23607$$

The difference occurs in the 4th decimal place.

Exercises 2.1

1. If $3000 is invested at 4.5% per annum for 5 years, find the amount at the end of the period if the interest is compounded:

(a) Annually

9

.

(b) Quarterly

(c) Monthly

(d) Daily

2. If $3000 is invested for 5 years, the interest being compounded monthly, compute the principal at the end of the period if the annual interest rate is:

(a) 4.5%

(b) 4.75%

(c) 5%

(d) 5.25%

(e) 5.5%

3. A 4-year certificate account pays 8.5% per year compounded quarterly.

   (a) If $8000 is deposited in the 4-year certificate account, what will be the principal at the end of the 4-year period?

   (b) What is the percentage increase of the principal at the end of the period over the original amount?

4. There was a TV commercial that ran something like this:

> When Suzie was born, her grandmother gave her $5000, which when kept in our account, will grow to $15,750 by the time Suzie reaches the age of 18, tripling the original amount.

   (a) Is that a good investment? Answer this question by computing the principals at the end of the 18th year at the following rates and also considering the current rates:

      (i) 4.5% per year compounded monthly.

(ii) 5.0% per year compounded monthly.

(iii) 5.5% per year compounded monthly.

(iv) 6.0% per year compounded monthly.

(v) 6.5% per year compounded monthly.

(b) Compute the annual interest rate used in the commercial, assuming that the interest was compounded monthly.

5. Benjamin Franklin in his will left 1000 pounds to the town of Boston to be lent out to young apprentices at 5% interest. He predicted that if the fund was managed well, it would grow to 131,000 pounds in 100 years. He specified that 100,000 pounds of the amount to be spent on public works and the remaining 31,000 pounds to be lent out as before for another 100 years. At the end of 100 years from the reception of the gift in 1894, the fund had grown to 90,000 pounds, not 131,000 pounds as he predicted. Supposing that the managers of the fund kept 31,000 pounds to be lent out and that the fund did bring in the interest at the rate of 5% per year compounded monthly, compute the present size of the fund.

2.2 Derivation of compound interest formula

We now give the derivation of the compound interest formula for the case m = 4, that is, when the interest is compounded quarterly. From the way the formula will be derived, though, it will be clear how the derivation will go for the other cases. The derivation is not difficult; it only requires the knowledge of simple factorization and the computation of interest.

We follow exactly the same pattern in Example 1 of the preceding section. We merely replace 1000 by P_0 and the interest rate 0.035 by r. Then, we have, with the same notation as in the example,

(1) $P_{01} = P_0 + P_0(r)(\frac{r}{4}) = P_0(1+\frac{r}{4})$

(2) $P_{02} = P_{01} + P_{01}(r)(\frac{r}{4}) = P_{01}(1+\frac{r}{4})$

(3) $P_{03} = P_{02} + P_{02}(r)(\frac{r}{4}) = P_{02}(1+\frac{r}{4})$

(4) $P_1 = P_{04} = P_{03} + P_{03}(r)(\frac{r}{4}) = P_{03}(1+\frac{r}{4})$

.

In order to express P_1 in terms of P_0, we first replace P_{01} in (2) by the right-hand side of (1):

(5) $$P_{02} = [P_0(1+\tfrac{r}{4})](1+\tfrac{r}{4}) = P_0(1+\tfrac{r}{4})^2$$

We then replace P_{02} in (3) by the right-hand side of (5):

$$P_{03} = [P_0(1+\tfrac{r}{4})^2](1+\tfrac{r}{4}) = P_0(1+\tfrac{r}{4})^3$$

Finally, we replace P_{03} in (4) by the right-hand side of the above equation:

$$P_{04} = [P_0(1+\tfrac{r}{4})^3](1+\tfrac{r}{4}) = P_0(1+\tfrac{r}{4})^4$$

Thus, if P_0 is deposited at the interest rate of r per year compounded quarterly, the principal P_1 at the end of the 1st year is given by

(6) $$P_1 = P_0\left(1+\frac{r}{4}\right)^4$$

To obtain the principal at the end of the nth year, we let

P_2 = the principal at the end of the 2nd year,
P_3 = the principal at the end of the 3rd year,
P_4 = the principal at the end of the 4th year,
and so on.

The computation of P_2 is the same as the computation of P_1, the only difference being that we begin with P_1 instead of P_0. So, by (6) we have

(7) $$P_2 = P_1\left(1+\frac{r}{4}\right)^4$$

Similarly,

(8) $$P_3 = P_2\left(1+\frac{r}{4}\right)^4$$

(9) $$P_4 = P_3\left(1+\frac{r}{4}\right)^4$$

(10) $$P_5 = P_4\left(1+\frac{r}{4}\right)^4$$

and so on.

We now replace P_1 in (7) by the right-hand side of (6):

(11) $$P_2 = \left[P_0\left(1+\frac{r}{4}\right)^4\right]\left(1+\frac{r}{4}\right)^4 = P_0\left(1+\frac{r}{4}\right)^8$$

Next we replace P_2 in (8) by the right-hand side of (11):

(12) $$P_3 = \left[P_0\left(1+\frac{r}{4}\right)^8\right]\left(1+\frac{r}{4}\right)^4 = P_0\left(1+\frac{r}{4}\right)^{12}$$

We continue this process. We can see that we will have

$$P_4 = P_0\left(1+\frac{r}{4}\right)^{16}$$

$$P_5 = P_0\left(1+\frac{r}{4}\right)^{20}$$

$$P_6 = P_0\left(1+\frac{r}{4}\right)^{24}$$

and so on. Thus, in general,

$$P_n = P_0\left(1+\frac{r}{4}\right)^{4(n)}$$

We summarize: If P_0 dollars is invested at the interest rate of r per annum compounded **quarterly**, the principal P_n at the end of the **nth year** is given by

$$P_n = P_0\left(1+\frac{r}{4}\right)^{4(n)}$$

In general, if P_0 dollars is invested at the interest rate of r per annum compounded **m times a year**, the principal P_n at the end of the **nth year** is given by

.

$$P_n = P_0\left(1+\frac{r}{m}\right)^{m(n)}$$

2.3 Installment purchase

We now give the formula by which we can compute the outstanding sum we owe on a loan for which we make monthly payments. We will give the principle behind the formula and that is used to obtain the formula. In this section we illustrate the use of the formula and how we can obtain another useful formula.

When we borrow P_0 dollars at the interest rate of r per year and agree to pay M dollars per month, the principal P_k we still owe at the end of the kth month is given by the formula

(1) $$P_k = P_0\left(1+\frac{r}{12}\right)^k - \frac{M\left[\left(1+\frac{r}{12}\right)^k - 1\right]}{\frac{r}{12}}$$

From this formula we can obtain another useful formula. When we borrow P_0 dollars at the interest rate of r per year and we want to pay off the loan in n month, we wan to find M for which $P_n = 0$. Then, we have to solve the equation

$$0 = P_0\left(1+\frac{r}{12}\right)^n - \frac{M\left[\left(1+\frac{r}{12}\right)^n - 1\right]}{\frac{r}{12}}$$

for M. When we do that, we get

(2) $$M = \frac{P_0\left(1+\frac{r}{12}\right)^n \cdot \frac{r}{12}}{\left(1+\frac{r}{12}\right)^n - 1}$$

This is the monthly installment formula by which we can compute the monthly payment that we have to make in order to pay off the loan of P_0 dollars at the interest rate of r per year in **n months**.

Example 1: If we buy a new $18,000 car and finance it with a loan at the interest rate of 6.25% per annum for 5 years to pay back in monthly installments, what is the monthly payment?

Solution: $$M = \frac{18000\left(1+\frac{0.0625}{12}\right)^{60} \cdot \left(\frac{0.0625}{12}\right)}{\left(1+\frac{0.0625}{12}\right)^{60} - 1} = 350.08671$$

Therefore, the monthly payment is $350.09.

Example 2: Miyanos want to buy a house , and they want to know an estimate of the monthly payment. They figure that they have to borrow $400,000 at the current 30-year mortgage rate of 5.75% per annum. What is their expected monthly payment?

Solution: $$M = \frac{400000\left(1+\frac{0.0575}{12}\right)^{360} \cdot \left(\frac{0.0575}{12}\right)}{\left(1+\frac{0.0575}{12}\right)^{360} - 1} = 2334.29142$$

So, their expected monthly payment is $2334.29. Of course, they have to buy a fire insurance and pay the real property tax, which the bank usually includes in the monthly payment. So, their monthly payment is somewhat larger than $2334.29.

When the monthly payment is not a nice number, the bank often requires the customer to pay a little more each month so that the monthly payment becomes a nice even figure. In the case of Example 1, it might be $351 per month instead of $350.09. When this is done, the difference must be adjusted at the final payment. We state the problem more precisely.

Example 3: If we borrowed \$18,000 at the interest rate of 6.25% per annum for 60 months to pay, and if the bank asked us to pay \$351 per month, what should the final payment (that is, the 60th payment) be?

Solution: We first have to compute P_{59}, the amount we still owe at the end of the 59th month. We use a modified form of Equation (1) of this section since the modified form is simpler in computation. The modified form looks like this:

$$P_k = P_0\left(1+\frac{r}{12}\right)^k - M\left[\left(1+\frac{r}{12}\right)^k - 1\right]\cdot\left(\frac{12}{r}\right)$$

Using this modified form, we have

$$P_{59} = 18000\left(1+\frac{0.0625}{12}\right)^{59} - (351)\left[\left(1+\frac{0.0625}{12}\right)^{59} - 1\right]\cdot\frac{12}{0.0625}$$

$$= 285.382315$$

So, we still owe \$285.38 at the end of the 59th month. The final payment is this amount plus the interest on this amount for one month:

$$\text{Final payment} = 285.38 + 285.38(0.0625)\left(\frac{1}{12}\right) = 286.866354$$

Therefore, the final payment should be \$286.87.

Exercises 2.3

1. Suppose that we buy a new car that costs \$15,000 and finance it with a loan at an interest rate of 6.5% per annum, the length of the loan being 5 years. If we pay \$294 per month, compute the amount we still owe at the end of the

   (a) first year.

   (b) second year.

   (c) third year.

   (d) fourth year.

   (e) 59th month.

2. Suppose that we buy a new car that costs $15,000 and we want to pay for it in 5 years. Compute the monthly payment if the annual interest rate is

   (a) 4.5%

   (b) 4.75%

   (c) 5%

   (d) 5.5%

   (e) 6%

   (f) 6.5%

   (g) 7%

3. Tom just bought a stereo set that cost $4000 with an "Easy Payment Plan". Under this plan, the interest rate was 18% per year and he could decide on the amount of the monthly payments.

   (a) If he decided to pay $100 per month, what will be the amount he would still owe at the end of

      (i) first year?

      (ii) second year?

      (iii) third year?

   (b) If he paid $100 per month, what would be the amount of interest he would pay on the

      (i) first year?

      (ii) second year?

4. A newspaper advertisement says that a $20,000 car can be had with 10% down and the rest can be financed at 6.75% per annum with 5 years to pay.

   (a) Compute the expected monthly payment if one accepts the deal.

   (b) If the bank requires the monthly payment of $355 instead of the amount computed in (a), find the final payment.

5. When we buy a used car, some financial institutions finance only 90% of the cost. Suppose that we decide to buy a used $9,000 car under this arrangement and secure a loan at 6.5% per annum. The length of the loan is 4 years.

   (a) What is our expected monthly payment?

   (b) If we make the monthly payment as computed in (a), what will be the amount we still owe at the end of

      (i) the first year?

      (ii) the second year?

      (iii) the third year?

6. Youngs bought their house in January of 1980 at $80,000 and took a loan at 9.5% per annum for it. The length of the loan was 30 years. The house is now valued at $340,000. They want to sell the house and buy a new one that will cost about $450,000.

(a) If they took the loan of $80,000 at 9.5% per annum to buy their house, what was the monthly payment?

(b) If they have been making the monthly payment computed in (a) for all these years, and this is May of 2008, what is the amount that they still owe on the original loan?

(c) When they sell the house, they have to pay the realtor 6% of the sale price. Suppose they can sell the house at $340,000, what is the amount that they can use for the down-payment of the new house?

(d) If the current 30-year mortgage rate is 6.85% per annum, what will be the monthly payment if they buy a house costing $450,000?