2.4 Derivation of the installment purchase formula

We now want to derive the formula by which we can compute the outstanding sum we owe on a loan for which payments are made in monthly installments. Before we do so, however, we need a simple formula to sum an expression of the form

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There are various ways of obtaining such a formula, but perhaps the simplest is to use the expansion of products. We know from elementary algebra (or we can simply compute) that

$$(t-1)(t+1) = t^2 - 1$$

$$(t-1)(t^2+t+1) = t^3 - 1$$

$$(t-1)(t^3+t^2+t+1) = t^4 - 1$$

$$(t-1)(t^4+t^3+t^2+t+1) = t^5 - 1$$

and so on. In fact, we can easily verify that

$$(t-1)(t^7+t^6+t^5+t^4+t^3+t^2+t+1) = t^8 - 1$$

which, by the way, is true for all values of t. Dividing both sides of the equation by (t – 1), we get

$$t^7+t^6+t^5+t^4+t^3+t^2+t+1 = \frac{t^8-1}{t-1}$$

From this formula, it is not difficult to see that

$$t^8+t^7+t^6+t^5+t^4+t^3+t^2+t+1 = \frac{t^9-1}{t-1}$$

$$t^9+t^8+t^7+t^6+t^5+t^4+t^3+t^2+t+1 = \frac{t^{10}-1}{t-1}$$

$$t^{10}+t^9+t^8+t^7+t^6+t^5+t^4+t^3+t^2+t+1 = \frac{t^{11}-1}{t-1}$$

and so on. In general, we have

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(1)

We are now in a position to derive the installment purchase formula. The formula is obtained for the following situation: Suppose we borrow dollars, to buy a car for example, at an annual interest rate of r and want to pay off the loan in n months, by making M dollars per month. We want to know how much we still owe at the end of the kth month (after the kth payment has been made). We assume that the payment begins exactly one month after the agreement is signed and each monthly payment is made on the same day of the month thereafter.

Let us look at an example to see exactly how the system works. Suppose that we borrow \$8,000 to buy a new car at the annual interest rate of 12.5% ($r = 0.125$). The length of the loan is 48 months. It turns out that the monthly payment is \$212.64. Let be the amount we still owe at the end of the kth month (after the kth payment has been made). When the monthly payment is made, the bank deducts the interest for one month on the amount we still owe, and the remainder is credited to the payment of the loan. When we make the first payment of \$212.64, the bank deducts the interest on \$8,000 for one month from it and the remainder goes to the payment of the loan. So,

For the second month, the interest is on this amount. Therefore,

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And so on.

We are now ready to derive the general formula using the following notation:

= the original amount of the loan,

r = the annual interest rate in decimal,

M = the amount of monthly payment,

= the outstanding principal at the end of the kth month.

To derive the formula for , we merely replace 8000 by , 212.64 by M, and 0.125 by r in the above computation, and we obtain

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The computation of is almost exactly the same as that of , except that we begin with instead of . Hence, we have

Similarly,

and so on.

To simplify the notation, we $t = 1 + \frac{r}{12}$. Then, the above equations become

Etc.

As in the case of the derivation of the compound interest formula, we want to express , , , etc., in terms of . Then,

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Now we see the pattern, and we have

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the last equality coming from Equation 5.2. We see, in general, that

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Substituting back $1+\frac{r}{12}$ for t,

$$P_k = P_0\left(1+\frac{r}{12}\right)^k - M\cdot\frac{\left(1+\frac{r}{12}\right)^k - 1}{\left(1+\frac{r}{12}\right) - 1}$$

Simplifying the denominator of the second term, we finally obtain

(2)

For computational convenience, we write the formula as

(3)

2.5 Savings program

Suppose we start the following savings program:

> We deposit M dollars every month (at the end of each month) into a savings account that pays the interest of r per year compounded monthly, Suppose further that we already have P_0 dollars in the savings account. Then, the amount P_n we have at the end of the nth month is given by the formula

(1) $$P_n = P_0\left(1+\frac{r}{12}\right)^n + \frac{M\left[\left(1+\frac{r}{12}\right)^n - 1\right]}{\frac{r}{12}}$$

As in the case of the installment purchase formula, we give another form for computational simplicity:

(2) $$P_n = P_0\left(1+\frac{r}{12}\right)^n + M\left[\left(1+\frac{r}{12}\right)^n - 1\right]\cdot\left(\frac{12}{r}\right)$$

For example, if we initially have \$1000 and make a monthly deposit of \$30 into the savings account that pays 5.5% per annum compounded monthly, then at end of the 20th year (or 240 month), we will have

$$P_{240} = 1000\left(1+\frac{0.055}{12}\right)^{240} + 30\left[\left(1+\frac{0.055}{12}\right)^{240} - 1\right]\cdot\left(\frac{12}{0.055}\right)$$

$$= 16065.4474$$

or \$16,065.45.

Suppose we want to accumulate the capital of \$100,000 in 20 years (assuming we initially have \$1000 and the interest rate is 5.5% per annum compounded monthly). We first solve Equation (1) for M and obtain

(2) $$M = \frac{\left[P_n - P_0\left(1+\frac{r}{12}\right)^n\right]\cdot\left(\frac{r}{12}\right)}{\left(1+\frac{r}{12}\right)^n - 1}$$

By substituting the data, we have

$$M = \frac{\left[100{,}000 - 1000\left(1+\frac{0.055}{12}\right)^{240}\right]\cdot\left(\frac{0.055}{12}\right)}{\left(1+\frac{0.055}{12}\right)^{240} - 1} = 222.6751017$$

Thus, it takes a monthly deposit of \$222.68 to accumulate the capital of \$100,000 in 20 years.

Exercises 2.5

1. If you open a savings account that pays 5.5% per annum compounded monthly and deposit \$50 per month, how much will you have at the end of the 10^{th} year?

2. Find the monthly deposit needed to accumulate the capital of \$20,000 in 10 years if the money is to be deposited in a savings account that pays 6% per annum compounded monthly and if the savings account initially has \$1,500.

3. Suppose one wants to buy a house that is in the range of \$450,000. He has to accumulate enough capital to make the down-payment, which is usually 20% of the cost of the house. If he wants to do it in 5 years, how much monthly savings does he have to make, assuming the interest rate is 5% per year compounded monthly?

4. A man bought a land for \$20,000 and made monthly payments of \$300 for 10 years. At the end of the tenth year the real estate salesman who sold the land offered to buy back the land for \$50,000, which the man accepted. Comparing his investment with depositing \$300 per month into a savings account that pays 6% per annum compounded monthly, was his investment on the land a good one?

5. During our life-time, most of us pay in the social security taxes at least \$200 per month for at least 35 years. If this amount were to be put into a bank at 6% per annum compounded monthly, what would be the amount at the end of the 35^{th} year?