Chapter 1 Properties of Numbers

We first review those properties of our number system that we use very frequently with or without realizing the specific properties. We will see how some computations can be simplified if we apply these properties judiciously. Even though computations will be done using a calculator, being able to estimate sums and products of numbers is still a very important skill that should never be neglected.

We note that in algebra we have basically only two operations --- addition and multiplication. What we call subtraction is addition by a negative number. Thus, when we write 20 – 6, we mean 20 + (-6). What we call division is multiplication by the reciprocal. For example, when we write $a \div b$, we mean $a\left(\frac{1}{b}\right)$ or $\frac{a}{b}$. Therefore, we list properties only for addition and multiplication.

1.1 Commutative and associative properties of addition

Commutative property of addition: a + b = b + a

Associative property of addition: (a + b) + c = a + (b + c)

These properties are usually applied together to simplify computations.

Example: Find the sum: $40 - 1 + 40 - 2 + 40 - 3 + 40 - 4 + 40 - 5$

Solution: The sum = $5(40) + (-1 - 2 - 3 - 4 - 5)$
= 200 – 15
= 185

Exercises 1.1

1. Compute the following:

(a) 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 =

(b) 51 + 52 + 53 + 54 + 55 + 56 + 57 + 58 + 59 =

(c) 30 – 5 + 30 – 4 + 30 – 3 + 30 – 2 + 30 – 1 + 30 – 9 + 30 – 8 + 30 – 7 + 30 – 6 =

(d) 60 – 1 + 60 – 2 + 60 – 3 + 60 – 4 + 60 – 5 + 60 + 1 + 60 + 2 + 60 + 3 + 60 + 4 =

2. Find the sum of the integers from 20 through 50, including 20 and 50.

3. Find the sum of the odd integers from 31 through 79.

1.2 Commutative and Associative Properties of Multiplication

Commutative property of multiplication: ab = ba

Associative properties of multiplication: a(bc) = (ab)c

These properties say that we can rearrange and regroup numbers in any way we want in computing a product of numbers. Here again we recall that dividing a number by a number n is the same as multiplying the number by the reciprocal of n, $\frac{1}{n}$. It is useful to know the following equivalences: $0.5=\frac{1}{2}$, $0.25=\frac{1}{4}$, $0.75=\frac{3}{4}$, and so on. Thus, instead of multiplying by 0.25, we can divide by 4 like the following:

$$0.25(84)=\frac{1}{4}(84)=21$$

Exercises 1.2

1. Compute the following:

(a) 5(7)(2)(9) =

(b) 3(4)(25)(27) =

(c) 0.25(48) =

(d) 0.75(36) =

2. How many nickels are in a two-dollar roll of nickels?

3. How many quarters are in a five-dollar roll of quarters?

1.3 Distributive Properties

$a(b + c) = ab + ac$ or

$(b + c)a = ba + ca$

We illustrate the application of this very useful property by the following examples.

Example 1: Compute 16(3.95).

Solution:

$$\begin{aligned} 16(3.95) &= 16(4 - 0.05) \\ &= 16(4) - 16(0.05) \\ &= 64 - 0.8 \\ &= 63.2 \end{aligned}$$

Example 2: Compute 6.75(12).

Solution:

$$\begin{aligned} 6.75(12) &= (7 - 0.25)(12) \\ &= \left(7 - \frac{1}{4}\right)(12) \\ &= 7(12) - \frac{1}{4}(12) \\ &= 84 - 3 \\ &= 81 \end{aligned}$$

Example 3: Simplify: $P - 0.8P$.

Solution: $P - 0.8P = P(1 - 0.8) = P(0.2) = 0.2P$

Exercises 1.3

1. Compute the following:

(a) 4.99(15) =

(b) 6.25(36)=

(c) 4.75(48) =

(d) 9.5(72) =

2. A box contains 36 candy bars, each costing $1.25. What is the total cost of the candy bars in the box?

3. (a) At a restaurant the price of a dinner is listed as $19.95 per person. If you have a party of 24 persons, what will be the total cost for the dinner?

(b) If you leave 15% tip, what is the amount of the tip?

1.4 Egyptian's Multiplication

Ancient Egyptians did not have the system of multiplication as we know today. They could only double a number or halve a number. They utilized this ability to devise a rather elaborate method of multiplication of numbers. The process is simple if the numbers are powers of 2, like 4, 8, 16, 32, 64, and so on. The multiplication of two numbers was carried out by halving the first number and doubling the second number and continuing this process until the first number became 1. For example, to compute the product 16(64), they did the following:

16(64)
8(128)
4(256)
2(512)
1(1024) = 1024

The number in the second line is clearly equal to the number in the first line because what has been done is $\left(\frac{1}{2}(16)\right)(2(64))$, and if we rearrange the numbers in the product, we get back to the original product since $\frac{1}{2}$ and 2 cancel out. So, each successive line is equal to the preceding line, and so $16(64) = 1024$.

They extended the method to computing the product of any whole numbers. But in this case, they had to take note of the line in which the first number is not divisible by 2. To get the next line they simply discarded the remainder, taking note, however, that the first number was not divisible by 2. They continued this process until the first number became 1. To get the desired product, they added all the second numbers of the lines in which the first numbers were not divisible by 2, and this sum was the desired product. Let us take an example. Compute the product 18(23).

18(23)
$9(\underline{46})$
4(92)
2(184)
$1(\underline{368})$

The desired product = 46 + 368 = 414. So, the product (18)(23) is equal to 414.

Why does the method work? We can explain the reason with the modern language why this method works. It works because whenever the first number is an odd number, we express it as an even number plus 1, and we are actually applying the distributive property as follows:

$$\begin{aligned}
18(23) &= 9(46) \\
&= (8 + 1)(46) \\
&= 8(46) + 46 \\
&= 4(92) + 46 \\
&= 2(184) + 46 \\
&= 1(368) + 46 \\
&= 368 + 46 = 414
\end{aligned}$$

Notice that the last line is exactly the sum we used to compute the product 18(23).

Let us take another example. Compute the product (25)(35).

$$\begin{aligned}
&(25)(35) \\
&12(70) \\
&6(140) \\
&3(280) \\
&1(560)
\end{aligned}$$

(25)(35) = 35 + 280 + 560 = 875

Justification:
$$\begin{aligned}
(25)(35) &= (24 + 1)(35) \\
&= (24)(35) + 35 \\
&= (12)(70) + 35 \\
&= (6)(140) + 35 \\
&= 3(280) + 35 \\
&= (2 + 1)(280) + 35 \\
&= (2)(280) + 280 + 35 \\
&= (1)(560) + 280 + 35
\end{aligned}$$

The last line is the sum that was used to compute the product (25)(35).

It is clear from these examples that the method surely works.